Published by Institute of Physics Publishing for SISSA/ISAS

Received: *January 31, 2003*
Revised: *February 25, 2003*
Accepted: *April 3, 2003*

Compactifications of heterotic theory on non-Kähler complex manifolds, I

Katrin Becker
Department of Physics, University of Utah
Salt Lake City, UT 84112-0830, USA
E-mail: katrin@physics.utah.edu

Melanie Becker
Department of Physics, University of Maryland
College Park, MD 20742-4111, USA
E-mail: melanieb@physics.umd.edu

Keshav Dasgupta
Department of Physics, Varian Lab, Stanford University
Stanford CA 94305-4060, USA
E-mail: keshav@itp.stanford.edu

Paul S. Green
Department of Mathematics, University of Maryland
College Park, MD 20742-4111, USA
E-mail: psg@math.umd.edu

Abstract: We study new compactifications of the SO(32) heterotic string theory on compact complex non-Kähler manifolds. These manifolds have many interesting features like fewer moduli, torsional constraints, vanishing Euler character and vanishing first Chern class, which make the four-dimensional theory phenomenologically attractive. We take a particular compact example studied earlier and determine various geometrical properties of it. In particular we calculate the warp factor and study the sigma model description of strings propagating on these backgrounds. The anomaly cancellation condition and enhanced gauge symmetry are shown to arise naturally in this framework, if one considers the effect of singularities carefully. We then give a detailed mathematical analysis of these manifolds and construct a large class of them. The existence of a holomorphic (3,0) form is important for the construction. We clarify some of the topological properties of these manifolds and evaluate the Betti numbers. We also determine the superpotential and argue that the radial modulus of these manifolds can actually be stabilized.

Keywords: Superstrings and Heterotic Strings, Superstring Vacua, Supergravity Models.

Contents

1. Introduction and summary

Compactifications of string theory and $\mathcal{M}$-theory with non-vanishing background fluxes are of great interest and an active area of research because of their implications for particle phenomenology. In such compactifications a potential for the moduli fields is generated at string tree level, so that many of the moduli fields can actually be stabilized with a mechanism, that is simple enough to do actual computations. String theory is then able to make definite predictions for the coupling constants of the standard model, such as the pattern of quark and lepton masses and the size of the gauge hierarchy. From the phenomenological point of view, compactifications of the heterotic string with background fluxes are particularly interesting. However, once the fluxes are turned on, the internal manifold is no longer Kähler and has torsion. This type of string theory compactifications was first discussed in [1, 2] and [3] some time ago. Yet, not much is known about such models in the literature, neither from the mathematics point of view nor from the physics point of view. It is the purpose of this paper to partially fill this gap by studying compactifications of the SO(32) heterotic string on compact, complex, non-Kähler manifolds both from the mathematics and physics point of view. Very generally, the internal manifolds, that we consider have a vanishing first Chern class and a vanishing Euler character. The existence of such non-Kähler manifolds and their realization as compact, complex three-folds was more recently pointed out in [4], as they originate from the warped $\mathcal{M}$-theory background of [5] by using a set of U-duality transformations. This warping is important, as the absence of warping will bring us back to Calabi-Yau (CY) type compactifications, where the three-form flux vanishes. After turning on the fluxes, these manifolds become inherently non-Kähler. This is to be contrasted with the non-Kähler four-folds, that we get in $\mathcal{M}$-theory with G-fluxes [5]. These four-folds are conformally CY, with the conformal factor being the warp factor. The three-folds, that we get in the heterotic theory are, in general, *not* conformally CY, as one cannot extract a conformal factor to obtain a Ricci-flat CY manifold. In fact the models presented in [4] and [6], consist of a four-dimensional CY base, that is warped in some specific way by the warp factor and a fiber over this base, that remains unwarped. However, there is a non-trivial *twisting*, which mixes the fiber coordinates with the complex coordinates of the base. This twist of the fiber and the warp factor of the base is responsible for making the manifold non Ricci-flat. Compactifications of SO(32) heterotic theory on these six-manifolds still give rise to a four-dimensional Minkowski vacuum [2, 1, 3], as the warp factor is just a function of the internal space and the three-form flux is non-vanishing only on the internal manifold. The three-form flux, which is real, is a *quantized* quantity, because it originates from quantized fluxes in $\mathcal{M}$-theory. It acts as the torsion for the manifold, in the presence of which, there is a preferred connection. All curvature related quantities are therefore measured with respect to this connection. There are also gauge bundles on the six-manifold, which are to be embedded in some specific way. As we will point out, the simple embedding of the spin connection into the gauge connection is not allowed for this kind of compactification, even though the second Chern classes satisfy

$$c_2(R) = c_2(F) \tag{1.1}$$

where R and F are the curvatures of the spin connection and the gauge connection respectively. In the above equation, somewhat unexpectedly, as we will show, the choice of connection is rather irrelevant. In the concrete example recently presented in [6], one can explicitly derive many of these facts, because the background is written is terms of an orbifold base, as we will discuss in this paper. However it is important to understand whether we can extend this example to the more general case, where we consider the base to be away from the orbifold limit. In this paper we shall present a detailed mathematical construction of a large class of complex non-Kähler manifolds, compute their Betti numbers and show, that the model of [6] is a special case of this general construction. From the physics point of view we will present the sigma model description of strings propagating in such backgrounds and study the effect of gauge fluxes, that were not taken into account in [6]. In particular, we show how the anomaly cancellation condition and enhanced gauge symmetry naturally arise in this framework, if one considers carefully the effect of singularities. Finally, we determine the form of the superpotential for compactifications of the heterotic string on non-Kähler manifolds and show, that the radial modulus can be stabilized in this type of compactifications.

Organization of the paper. This paper is organized as follows. In section 2 we discuss the geometrical properties of non-Kähler manifolds. In sub-section 2.1 we recapitulate the concrete heterotic background described in [6], where the gauge fluxes originating from the orbifold singularities of the internal manifold had not been taken into account. In sub-section 2.2 we find the explicit solution to the warp factor equation, ignoring the localized fluxes coming from the orbifold singularities. In sub-section 2.3 we describe the torsional connection appearing in the model of [6] and show, that this model has vanishing first Chern class, by taking the gauge fields into account. In sub-section 2.4 we recapitulate the sigma model description and use this description to show, that the internal manifolds considered herein are not Kähler. In sub-section 2.5 we study the effects of the gauge fields and show how the anomaly relation of the heterotic theory originates from the type-IIB theory. In sub-section 2.6 we show, how the full non-abelian symmetry originates from the $\mathcal{M}$-theory dual description.

In section 3 we discuss the mathematical aspects of non-Kähler manifolds, in particular their topological properties. We present various algebraic geometric properties of these manifolds and give a generic construction of a large class of compact, complex non-Kähler manifolds. In sub-section 3.1 we first recapitulate some basic features of non-Kähler manifolds, which might be lesser known to the reader. As a special case of the Serre spectral sequence we consider the case of a torus bundle over a base manifold B. If B is a simply-connected four-dimensional manifold, we can easily compute the value of the Betti numbers and show that the Euler characteristic of these non-Kähler manifolds vanishes. We use this general result to compute the Betti numbers of the model considered in [6]. In sub-section 3.2 we give a generic construction of non-Kähler manifolds, that are generalizations of the Hopf surface, which is a torus bundle over CP^1 and show, that the model considered in [6] is a special case of this general construction.

In section 4 we compute the background superpotential for compactifications of the heterotic string on a non-Kähler manifold by dimensional reduction of the quadratic term appearing in the action of $\mathcal{N} = 1$, $D = 10$ supergravity coupled to Yang-Mills theory. We also use T-duality applied to the type-IIB superpotential, that is well known in the literature, to get the form of the heterotic superpotential. In sub-section 4.1 we discuss the existence of a unique holomorphic three-form for the particular model of type-IIB theory compactified on $T^4/\mathcal{I}_4 \times T^2/\mathbb{Z}_2$ and interpret the norm of the holomorphic three-form in terms of the torsion classes presented in [7, 8]. In sub-section 4.2 we carry out the explicit calculation of the superpotential. In sub-section 4.3 we shall see, that many of the moduli fields for the heterotic compactification are lifted by switching on the $\mathcal{H}$-fluxes. In sub-section 4.4 we compute the potential for the radius moduli and show, that the value of this field can actually be fixed.

JHEP04(2003)007

Finally, in section 5 we present our conclusions and discuss some related examples of non-Kähler manifolds that have recently appeared in the literature. Some of these examples can be constructed from the generic structure in section 3. We also point out some directions for phenomenological applications and elaborate some implications of our results that could be pursued in near future.

Note added: while the draft was being written, there appeared a couple of papers, which have some overlap with the contents of this paper. The paper of [7] discusses a particular kind of non-Kähler manifold called the Iwasawa manifold. They also classified the torsion classes. More discussions on this classification also appeared in [8]. Detailed mathematical aspects of non-Kähler manifolds with SU(3) structure in the heterotic theory were given in [9]. While this paper was being written we became aware, that an independent calculation of the superpotential for the heterotic string on a manifold with torsion was being performed (see the second reference in [10]). We thank the authors for informing us about their calculation prior to publication.

2. Geometrical properties of non-Kähler manifolds

In the previous paper [6] two of us gave an explicit example of an $\mathrm{SO}(8)^4$ heterotic theory compactified on a non-Kähler manifold with non-vanishing Ricci tensor and a background $\mathcal{H}$-field. It was shown, how this background satisfies the torsional constraints expected from supersymmetry. This analysis was mainly motivated from the existence of a solvable warped background [5] in $\mathcal{M}$-theory with G-fluxes on the $T^4/\mathcal{I}_4 \times T^4/\mathcal{I}_4$ manifold, where $\mathcal{I}_4$ is a purely orbifold action, which reverses the sign of all the toroidal directions. We denote the unwarped metric of the four-fold by $\tilde{g}_{a\bar{b}}$. The G-flux, that we consider is

$$\frac{G}{2\pi} = A\ d\bar{z}^1 dz^2 d\bar{z}^3 dz^4 + B\ dz^1 d\bar{z}^2 d\bar{z}^3 dz^4 + \sum_{i=1}^{4} F^i \wedge \Omega^i + \text{c.c.} \tag{2.1}$$

Here $z's$ describe the coordinates on the internal manifold, Ω^i with $i = 1, \ldots, 4$ are the harmonic two forms on $T^4/\mathcal{I}_4$, F^i are the $\mathrm{SO}(8)^4$ gauge bundles at the singular orbifold points of $T^4/\mathcal{I}_4$ with an SO(8) gauge bundle at each point and A,B are complex constants, whose explicit value is given later on. By construction (2.1) is a primitive $(2,2)$ form and therefore preserves supersymmetry. In fact, if we ignore the localized piece in (2.1), then one can easily show that in the type-IIB theory we get three form fluxes $H_{NS} = H$ and $H_{RR} = H'$ which, when combined[1] to form G_3,

$$G_3 = H' - \varphi H = 2\,i\mathrm{Im}\,\varphi \left(\bar{A}\ dz^1 \wedge d\bar{z}^2 \wedge dz^3 + \bar{B}\, d\bar{z}^1 \wedge dz^2 \wedge dz^3\right), \tag{2.2}$$

clearly indicates, that only the $(2,1)$ piece survives. Here φ is the usual axion-dilaton combination in type-IIB. This is consistent with the constraints imposed by supersymmetry, as these predict the vanishing of the $(3,0), (0,3)$ and $(1,2)$ parts of G_3. Also notice, that

[1] Our conventions are slightly different from [6], as we take $\mathcal{T}$ as our torsion and not $-\mathcal{T}$. Therefore, our definition of G_3 will be the same as in [11].

the background previously studied in [6] can actually be mapped to this one by an $\mathrm{SL}(2,\mathbb{Z})$ matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{2.3}$$

which has a fixed point at $\varphi = i$. Furthermore, the definition of G_3 in (2.2) can be inferred directly from $\mathcal{M}$-theory. If we define a generic (1,2) form as ω, then the G-flux in $\mathcal{M}$-theory is

$$G = \omega \wedge dz^4 - \omega^* \wedge d\bar{z}^4 = \frac{\bar{G}_3}{\varphi - \bar{\varphi}} \wedge dz^4 - \frac{G_3}{\varphi - \bar{\varphi}} \wedge d\bar{z}^4 \,, \tag{2.4}$$

where $dz^4 = dx^{10} + \varphi dx^{11}$. Now wedging this with a holomorphic (4,0) form and using the relation: $\int dz^4 \wedge d\bar{z}^4 = 2i \ \mathrm{Im}\varphi \ dx^{10} \wedge dx^{11}$, we can easily reproduce the IIB superpotential $\int G_3 \wedge \Omega$ and hence infer G_3 from there.

As discussed in [6], the localized flux in (2.1) which involve the F^i fields contribute to the world volume gauge fluxes and the non-localized fluxes are responsible for the $\mathcal{H}$-torsion and the geometry in the heterotic description. However, the analysis of [6] took mostly the non-localized fluxes into account and ignored more or less the localized fluxes. This led to

$$d\mathcal{H} = 0 \,, \tag{2.5}$$

as the condition on the $\mathcal{H}$ field. However, due to supersymmetry and anomaly cancellation, it is well known that $\mathcal{H}$ cannot be a closed form. It was noted in [6], that the localized fluxes, when taken into account, should give the right anomaly condition on the heterotic side. In this section, among other things, we will verify this fact. The constraint on the $\mathcal{H}$-flux will be

$$d\mathcal{H} = \frac{\alpha'}{2}[p_1(R) - p_1(F)] \,, \tag{2.6}$$

where p_i are the Pontryagin classes for the spin and gauge bundles respectively. Since $\mathcal{H}$ and $J_{a\bar{b}}$ (the fundamental form) are related by the torsional constraints [2, 1, 3], the fundamental form is very constrained because of this and non-Kählerity. These issues have been already discussed in [6], where it was shown, how the fundamental form satisfies the torsional constraints *ignoring* the gauge fluxes. In this paper we shall take these into account and study their effects.

2.1 The background geometry

Let us begin by writing the background geometry explicitly. The notations used here are the same as in [6]. We shall also take the type-IIB coupling $g_B = 1$. The heterotic background is:

$$\begin{aligned} ds^2 &= 4\Delta^2 \ \tilde{g}_{1\bar{1}}\tilde{g}_{3\bar{3}} \ dz^1 d\bar{z}^1 + 4\Delta^2 \ \tilde{g}_{2\bar{2}}\tilde{g}_{3\bar{3}} \ dz^2 d\bar{z}^2 + |dz^3 + 2B\bar{z}^2 dz^1 - 2A\bar{z}^1 dz^2|^2 \,, \\ \mathcal{H} &= A \ dz^2 \wedge d\bar{z}^1 \wedge d\bar{z}^3 - B dz^1 \wedge d\bar{z}^2 \wedge d\bar{z}^3 + \bar{A} \ d\bar{z}^2 \wedge dz^1 \wedge dz^3 - \bar{B} \ d\bar{z}^1 \wedge dz^2 \wedge dz^3 \,, \\ g_{\mathrm{het}} &= 2\Delta \ \tilde{g}_{3\bar{3}} \,, \end{aligned} \tag{2.7}$$

where Δ is the warp factor and g_{het} is the heterotic coupling constant and the constant values of A and B are given below. The three-form background $\mathcal{H}$ is a *real* quantity which satisfy the torsional constraints [6]. The unwarped metric $\tilde{g}_{ij}$ is basically flat everywhere except at some points, where there are singularities. In fact in $\mathcal{M}$-theory the contributions to the X_8 polynomial come from those singular points. Ignoring them will lead us to trivial background fluxes. Taking this into account, the metric can be simplified to

$$g_{a\bar{b}} = \begin{pmatrix} \Delta^2 + |Bz^2|^2 & -ABz^1\bar{z}^2 & \frac{B\bar{z}^2}{2} \\ -A\bar{B}\bar{z}^1 z^2 & \Delta^2 + |Az^1|^2 & -\frac{A\bar{z}^1}{2} \\ \frac{\bar{B}z^2}{2} & -\frac{\bar{A}z^1}{2} & \frac{1}{4} \end{pmatrix}, \tag{2.8}$$

where we have ignored an overall factor of 4. In the presence of localized fluxes the metric will be more involved, as we shall derive towards the end of this section. We are also assuming, that the warp factor is independent of the z_3 direction. In the following we shall show that

$$\Delta \equiv \Delta(|z_1|, |z_2|), \tag{2.9}$$

is a consistent assumption. From (2.7) we see that the base, with coordinates z^1, z^2 is stretched by $\Delta^2 \tilde{g}_{3\bar{3}}$ and the torus parametrized by z^3 is non trivially fibred over the base. The above metric can be generated from $\mathcal{M}$-theory on a four-fold by making a series of U-duality transformations. If we denote the volume of any typical four-cycle of the four-fold by v, then the constants A, B appearing in (2.7) have definite values of

$$A = \frac{2+i}{v}, \qquad \text{and} \qquad B = \frac{i}{v}. \tag{2.10}$$

In deriving this we have to be careful about the total volume of our space.[2] As it turns out (and is also discussed in [6]) the volume of our space is reduced by 1/4 and hence all the quantization rules are changed slightly. Furthermore, notice that the previous values for A and B differ by a factor $\frac{1}{v}$ from the results of [6], where v was set to one for convenience. The above background is not complete, unless we specify the gauge bundle. The gauge bundle for our case is $D_4^4 = \mathrm{SO}(8)^4$ which, as discussed in [6], comes from stabilizing the seven brane moduli. This fixing also guarantees, that there are no non-perturbative corrections to the moduli space.

2.2 Solution to the warp factor equation

The solution to the warp factor equation can be found in two stages. We will attack this question from the type-IIB point of view, where we have the advantage of writing the warp factor equation linearly. Therefore, let us first assume our manifold is *non-compact.* In this way we can concentrate on regions far from the orbifold singularities and write the warp factor equation as

$$\frac{\partial \Delta^2}{\partial z^2} + |B|^2 \bar{z}^2 = \frac{\partial \Delta^2}{\partial z^1} + |A|^2 \bar{z}^1 = 0. \tag{2.11}$$

As discussed in [6], the above equations have an origin on the type-IIB side from the self-duality relation for the five-form F_5. The solution to (2.11) is

$$\Delta^2 = \mathrm{c}_o - \frac{5\ |z^1|^2 + |z^2|^2}{v^2}, \tag{2.12}$$

[2]We do not necessarily mean that *all* four-cycles have equal volumes. The above analysis goes through easily with arbitrary sizes of the four-cycles.

where v is the volume of any generic four-cycle of the $\mathcal{M}$-theory four-fold. We will fix c_o below. Also observe, that the warp factor is independent of the z^3 direction and depends only on $|z^i|$, for $i = 1, 2$.

Let us now bring back the singularities by making the type-IIB manifold compact. The term, that will now contribute to the warp factor equation is the X_8 polynomial of $\mathcal{M}$-theory as

$$X_8 = \frac{3}{32} \sum_{z^i, w^j} \delta^4\left(z - z^i\right) \delta^4\left(w - w^j\right), \tag{2.13}$$

where $z^i = (z_i^1, z_i^2)$ and $w^j = (z_j^3, z_j^4)$ are the fixed points of $T^4/\mathcal{I}_4 \times T^4/\mathcal{I}_4$. This polynomial, which is a bulk term in $\mathcal{M}$-theory, will appear on the type-IIB side as gravitational couplings on the $D7$ branes and $O7$ planes and will give no contribution to the bulk. This will modify the type-IIB warp factor equation to

$$\Box\, \Delta^2 = -2(|A|^2 + |B|^2) + \sum_{i,j} \delta^2\left(w - w^i\right) \delta^4\left(z - z^j\right), \tag{2.14}$$

where the operator $\Box$ has been introduced in [6] and we have ignored the contributions to the warp factor from gauge bundles. Observe that (2.14) doesn't take the delocalization along $w \equiv z^3$ into account, because this is derived directly from (2.13). If we take this effect into account, the solution of the type-IIB warp factor equation becomes

$$\Delta^2 = \mathrm{c}_o - \frac{5\ |z^1|^2 + |z^2|^2}{v^2} + \sum_{a,b} \frac{\mathrm{c}_1}{|z^1 - a|^2\ |z^2 - b|^2}, \tag{2.15}$$

where a, b are the fixed points on the base and c_1 is defined below. The constant c_0 is the boundary value of the warp factor, i.e it's value, when the size of the compact manifold is very large. In other words,

$$\mathrm{c}_o = {}^{\lim}_{a,b,v\ \to\ \infty} \Delta^2, \tag{2.16}$$

and the manifold therefore becomes unwarped. Observe that the warp factor (2.15) is well behaved everywhere except at the orbifold points. This is because we have taken $\mathrm{tr}(R \wedge R) = \sum_j\ \delta^4(z - z^j)$. For a resolved orbifold (i.e for a $K3 \times K3$ compactification) the warp factor will be well defined everywhere. We can incorporate this by assuming

$$\mathrm{c}_1 \equiv \mathrm{f}\left(|z^1 - a|, |z^2 - b|\right) = 1, \qquad \text{and} \qquad \mathrm{f}(0,0) = 0, \tag{2.17}$$

where the first equality holds away from the fixed points. Similar arguments can be given, when we include the gauge fluxes in the warp factor equation, the form of which is given in [6]. In fact the warp factor is well defined everywhere because it is well defined on every *patches* of our manifold even though the two-form field B is not globally defined [6]. This also resolves another issue regarding the periodicity of the z^i coordinates. The form of the warp factor presented above in (2.15) doesn't seem to have the required periodicity. This is because we have defined the source of the warp factor (the two form B) on patches. Another point to note here is, that for a large sized manifold, even though the warp factor vanishes, there are still non-vanishing fluxes. The constant flux density goes to zero (because $A, B \to 0$ for $v \to \infty$) but the localized fluxes remain non-zero at the fixed points (which are shifted to infinity). This is where we face a contradiction, because the manifold seems to behave as a Calabi-Yau but there are still fluxes left, which cannot be supported on a Calabi-Yau. The resolution, as discussed in [1, 6], is to assume, that these manifolds do not have a large radius limit. We'll discuss more about this in section 4.4. Further details on the function f will be presented elsewhere.

2.3 The torsional connection and Chern-class

Another interesting question is the choice of connection for the compactifications considered herein. This is important because our manifolds will be complex with a covariantly constant complex structure J_j^i, i.e

$$\nabla_i J_k^j \equiv \partial_i J_k^j + \Omega_{il}^j \, J_k^l - \Omega_{ik}^l \, J_l^j = 0\,, \tag{2.18}$$

where Ω_{jk}^i is the connection used. We cannot use the Christoffel connection because, as we show below, this will lead to a contradiction. The connection, that will be consistent for us is the torsional connection defined as

$$\Omega_{jk}^i = \Gamma_{jk}^i \pm \mathcal{T}_{jk}^i\,, \tag{2.19}$$

where Γ_{jk}^i is the Christoffel connection and $\mathcal{T}_{ijk}$ is the torsion. The ambiguity of the sign will be clarified in the next sub-section. The torsion is related to the background $\mathcal{H}$ by the following relation:

$$\mathcal{T}_{ijk} = \mathcal{T}_{ij}^l \, g_{lk} = \frac{1}{2}\mathcal{H}_{ijk}\,. \tag{2.20}$$

As discussed in detail in [2, 1], and [6], the covariantly constant complex structure (2.18) implies the following important relation between the $\mathcal{H}$ field and the metric (in terms of complex coordinates)

$$\mathcal{H}_{ab\bar{c}} = 2g_{\bar{c}[a,b]}\,, \tag{2.21}$$

which is called the torsional constraint. In [6] we showed explicitly, how the background (2.7) satisfies the equations (2.21) (and its complex conjugates). In fact, the derivation of (2.21) from (2.18) requires the fact that the Nijenhaus tensor

$$N_{mnp} = \mathcal{H}_{mnp} - 3J_{[m}^q \, J_n^r \, \mathcal{H}_{p]qr}\,, \tag{2.22}$$

vanishes. This can be easily shown, by taking into account the dilatino supersymmetry equation for the heterotic background. Inserting (2.21) in (2.20), we can determine the connection we should use. For example, with the choice of minus sign in (2.19) it is given by a simple relation

$$\Omega_{bc}^a = g^{a\bar{d}} \, g_{\bar{d}c,b} = \bar{\Omega}_{\bar{b}\bar{c}}^{\bar{a}}\,, \tag{2.23}$$

with all other components zero. For our case the preferred connection will include a plus sign in (2.19), as we shall see later on. One can show, that for Kähler manifolds the above connection reduces to the Christoffel connection.[3]

[3]In the next section (section 3), where we discuss the mathematical properties of non-Kähler manifolds, we will refer to the torsional connection simply as ω. This will symbolize the *preferred* connection for our case. For the later sections we will distinguish between the affine connection ω_i^{ab} (or the Christoffel connection Γ_{ij}^k) and the torsion $\mathcal{H}_{ijk}$, unless mentioned otherwise.

JHEP04(2003)007

The above choice of the connection solves one of the apparent puzzles related to the form of the tr$R \wedge R$ term. As explained in [1], as a consequence of the anomaly cancellation condition the fundamental form of a manifold with torsion has to satisfy

$$i\partial\bar{\partial}J = \mathrm{tr}\,(F \wedge F - R \wedge R)\,. \tag{2.24}$$

In general, tr$R \wedge R$ will contain a $(3,1)$ and $(1,3)$ piece, if we use the Christoffel connection, giving in most cases Calabi-Yau manifolds as backgrounds. By using the connection given in (2.23), it is possible to show, that tr$R \wedge R$ will always be a $(2,2)$-form and in this case we expect to find a manifold with torsion as a solution.

Such a manifold with torsion will, in general, be non-Kähler. As has been explained in [2], if the vector

$$v^m = J^{mn} J^{kl} \mathcal{H}_{nkl}\,, \tag{2.25}$$

vanishes the manifold is semi-Kähler, while if $\mathcal{H}_{ijk}$ vanishes the manifold is Kähler. For the background geometry (2.7) it is easy to see, that

$$v_m = \partial_m (\log \det g)\,, \tag{2.26}$$

where det $g = \det g_{a\bar{b}} = \frac{1}{2}\Delta^4$. This implies a non-vanishing value for v^m and therefore the manifold defined in (2.7) is not even semi-Kähler.

It is also easy to show, that the manifold (2.7) has a vanishing first Chern class. Indeed the trace of the curvature 2-form is given by

$$\mathcal{R} = R_{mnkl} J^{kl} dx^m \wedge dx^n = i\partial\bar{\partial}(\log \det g) = d(\partial - \bar{\partial}) \parallel \Omega \parallel^2\,, \tag{2.27}$$

since the norm of the holomorphic $(3,0)$-form is given in terms of the warp factor. In order to get the above result, we have used the connection defined in (2.23), instead of the Christoffel connection. The first Chern class, given by $c_1 = \int \mathcal{R}$ vanishes, since Ω is globally defined. An alternative way to see this would be to use (2.18). Contracting (2.18) by J^n_m we get (in complex coordinates):

$$J^b_a \tilde{\nabla}_c J^c_b = \mathcal{H}^d{}_{ad} - \mathcal{H}^{\bar{d}}{}_{a\bar{d}} = \partial_a \phi\,, \tag{2.28}$$

where ϕ is the dilaton, $\tilde{\nabla}$ is the torsion free covariant derivative and the last equality comes from the supersymmetry transformation of the dilatino, as explained in [6]. The above expression is a total derivative.

2.4 Sigma model description

(a) **Basic concepts.** Let us now discuss the sigma model description of our model. Most of the details, which are well known, can be extracted from [12, 2, 1] and therefore we shall be brief. To the lowest order in α', the light-cone-gauge action for the heterotic string in the presence of a non-zero two-form potential B and a gauge field $F = dA$ is given by

$$\begin{aligned} S = \frac{1}{8\pi\alpha'} \int d\sigma d\tau \Big[& \partial_+ X^i \partial_- X^j (g_{ij} + B_{ij}) + iS^n (D_+ S)^n + i\Psi^A (D_- \Psi)^A + \\ & + \frac{1}{2} F_{ij\,AB} \sigma^{ij}_{mn} S^m S^n \Psi^A \Psi^B + \mathcal{O}(\alpha') \Big]\,, \end{aligned} \tag{2.29}$$

where $X^\mu = (X^+, X^-, X^i)$, $(i = 1, \dots, 8)$ are the bosonic fields and S^m with $m = 1, \dots, 8$, describe spinors in the vector representation of SO(8), which are the superpartners of X^i. The remaining degrees of freedom are the anti-commuting world-sheet spinors Ψ^A, $A = (1, \dots, 32)$, which transform as scalars under SO(8). On-shell we also impose

$$\partial_+ S^m = 0 = \partial_- \Psi^A \,, \tag{2.30}$$

so that S^m are right moving, while Ψ^A are left moving. If we denote the Yang-Mills field representations by T^q_{AB}, where q labels the adjoint of the gauge group, then $F_{ij\,AB} = F^q_{ij} T^q_{AB}$ (with a similar notation for the corresponding one-form potential A_i), while the anti-symmetrized product of gamma matrices is written as σ^{ij}_{mn}. Finally, the covariant derivatives appearing in (2.29) are given by

$$\begin{aligned}(D_-\Psi)^A &= \partial_-\Psi^A + A_i^{AB}\left(\partial_- X^i\right)\Psi^B \,, \\ (D_+ S)^m &= \partial_+ S^m + \frac{1}{2}\left(\omega_i^{ab} - \mathcal{T}_i^{ab}\right)\sigma^{mn}_{ab}(\partial_+ X^i) S^n \,.\end{aligned} \tag{2.31}$$

Observe the way the spin connection ω_i^{ab} and the torsion $\mathcal{T}_{ijk}$ appear in the covariant derivative of S^m. If $e_i^a(X)$ represent some orthonormal frame, such that $e_i^a e_j^b \delta_{ab} = g_{ij}$, then the spin connection satisfies the torsion *free* equation

$$de^a + \omega^{ab} \wedge e^b = 0 \,. \tag{2.32}$$

If we denote the local gauge rotations by α^{AB} (under which Ψ^A rotates) and the local SO(8) Lorentz-rotations by β^{mn} (under which S^m transforms), then in general these symmetries are anomalous. As is well known, these anomalies cancel, if we impose the constraint (2.6) on the $\mathcal{H}$ field. In fact the B field has to transform in the following way [13]

$$\delta B_{ij} = \alpha' \left[\gamma_{[i} \cdot \partial_{j]}\beta - A_{[i} \cdot \partial_{j]}\alpha\right] , \tag{2.33}$$

to cancel the anomalies. Here γ is used to define the curvature R as $R = d\gamma + \gamma \wedge \gamma$. In the next sub-section we will show how this is related to the usual spin connection. We will also discuss later how to actually realize this from our $\mathcal{M}$-theory set-up.

The background metric and the B field of the particular example constructed in [6] are already given in (2.7). In this case the unwarped metric is flat $\tilde{g}_{a\bar{b}} = \eta_{a\bar{b}}$. Using this we can calculate the inverse of the complete metric, $g^{a\bar{b}}$, and obtain

$$g^{a\bar{b}} = \frac{1}{\Delta^2}\begin{pmatrix} \frac{1}{2} & 0 & -B\bar{z}^2 \\ 0 & \frac{1}{2} & A\bar{z}^1 \\ -\bar{B}z^2 & \bar{A}z^1 & 2(\Delta^2 + |Az^1|^2 + |Bz^2|^2)\end{pmatrix} , \tag{2.34}$$

where Δ is as usual the warp factor. Taking the inverse of the metric and the expression for $\mathcal{H}$ given in (2.7), one can easily confirm

$$\mathcal{H}_{\bar{a}b\bar{c}} g^{b\bar{c}} = \partial_{\bar{a}}\phi \,. \tag{2.35}$$

This in turn is consistent with the fact that we have a vanishing first Chern-class, as discussed in the previous sub-section.

We should point out, that because of (2.33), which leads to $d\mathcal{H} \neq 0$ we cannot write the metric in terms of any *local* potential K because

$$g_{a[\bar{b},\bar{c}]d} - g_{d[\bar{b},\bar{c}]a} = \alpha' f(A,\gamma) \qquad \Rightarrow \qquad J = i(\partial\bar{K} - \bar{\partial}K) + \mathcal{O}(\alpha') \,. \tag{2.36}$$

Here $f(A,\gamma)$ is a function, that can be determined from (2.33). Therefore, the internal manifold is not Kähler. However, if $d\mathcal{H} = 0$, we can have a local potential in terms of which we can write the metric, as the function $f(A,\gamma)$ will be vanishing in this case. But, as briefly mentioned in [6], this will make the warp factor constant and therefore there will be no warped solution. A way to see this (at leading order in the α' expansion) is as follows. If we take the torsional connection into account, the supersymmetry variation of the gravitino can be written concisely as

$$\mathcal{D}_m\epsilon = \left(\partial\!\!\!/\phi - \frac{1}{6}\mathcal{H}\right)\epsilon = 0 \,. \tag{2.37}$$

After a few simple but cumbersome manipulations (2.37) gives rise to the following condition [3]

$$\frac{4}{3}\Gamma^{mnpq}\tilde{\nabla}_m\mathcal{H}_{npq}\epsilon = \left[R + \frac{16}{3}\mathcal{H}^2_{mnp} + 20(\partial_m\phi)^2 + 6\partial^m\partial_m\phi\right]\epsilon \,, \tag{2.38}$$

where $\tilde{\nabla}$ is the torsion free covariant derivative. Embedding the spin connection into the gauge connection makes the left hand side of (2.38) to vanish. The right hand side of (2.38) then takes the form

$$R + \frac{16}{3}\mathcal{H}^2_{mnp} + 20(\partial_m\phi)^2 + 6\partial^m\partial_m\phi = 0 \,, \tag{2.39}$$

which is consistent with the fact, that supersymmetry imposes the classical equation of motion (assuming, of course, that the transformation laws are on-shell). The contracted Einstein equation and the equation for the three-form field $\mathcal{H}$ come out directly from the supersymmetry transformations. In the absence of sources and singularities this equation can be integrated over the internal manifold giving the condition

$$\int d^6x\sqrt{g}\left[R + \frac{16}{3}\mathcal{H}^2_{mnp} + 14(\partial_m\phi)^2\right] = 0 \,. \tag{2.40}$$

If supersymmetry is unbroken, all the terms in the above sum are positive and therefore embedding the spin connection into the gauge connection implies (at this order in the α' expansion)

$$\mathcal{H}_{mnp} = 0 \qquad \text{and} \qquad \phi = \text{const} \,. \tag{2.41}$$

It then follows from (2.41) that

$$\tilde{\nabla}_m\epsilon = 0 \,, \tag{2.42}$$

which implies a Ricci-flat Kähler metric as our solution. An alternative proof based on the non-existence of a holomorphic $(3,0)$ form, when we make the assumption $d\mathcal{H} = 0$ is given in [14].

(b) **Anomaly cancellation.** Something discussed above in (2.41) and (2.42) may seem a little puzzling. Let us go back again to (2.31) and make the following identifications

$$A_i^{ab} \leftrightarrow \left(\omega_i^{ab} - \mathcal{T}_i^{ab}\right),$$
$$\Psi^A \leftrightarrow S^{\dot{p}}, \quad A = 1, \ldots, 8, \tag{2.43}$$

where the rest of the Ψ^A for $A = 9, \ldots, 32$ do not couple to the background fields and $\dot{p}$ labels the components of a spinor in the 8_c representation of SO(8). In fact, the above transformations can be used to rewrite our heterotic string sigma model lagrangian (2.29) with *all* the left-movers replaced in the following form

$$\int d\sigma d\tau \left[iS^{\dot{p}}(D_- S)^{\dot{p}} + \frac{1}{4}R_{ijkl}\sigma^{ij}_{\dot{p}\dot{q}}\sigma^{kl}_{mn}S^{\dot{p}}S^{\dot{q}}S^m S^n\right]. \tag{2.44}$$

In this form the above lagrangian resembles the Green-Schwarz superstring moving in a curved background. This theory is anomaly free, which would imply that it doesn't receive any Chern-Simons corrections. In the above action the curvature tensor R_{ijkl} is defined with respect to the connection

$$\tilde{\omega}_i^{ab} = \omega_i^{ab} - \mathcal{T}_i^{ab} \equiv \omega_i^{ab} - \frac{1}{2}\mathcal{H}_i^{ab}, \tag{2.45}$$

and contractions are done with e_i^a. As discussed by [2], there would, in general, be higher order α' corrections to (2.29), which in fact vanish, if we embed the gauge field in the torsional-spin connection $\tilde{\omega}_i^{ab}$, as done above. In other words, we identify γ in (2.33) as

$$\gamma_i^{ab} = \omega_i^{ab} - \frac{1}{2}\mathcal{H}_i^{ab}. \tag{2.46}$$

The three-form field strength, that is invariant under the transformation [13] is

$$\mathcal{H} = dB - \alpha'\left[\Omega_3(A) - \Omega_3\left(\omega - \frac{1}{2}\mathcal{H}\right)\right], \tag{2.47}$$

where we have defined

$$\Omega_3(A) = A \wedge F - \frac{1}{3}A \wedge A \wedge A, \tag{2.48}$$

and $\Omega_3(\omega - \frac{1}{2}\mathcal{H})$ is given by a similar equation with A replaced by $\omega - \frac{1}{2}\mathcal{H}$. The above $\mathcal{H}$ field satisfies the Bianchi identity (2.6). At this point it seems from the analysis done in the previous sub-section, that we would not have a warped solution because $d\mathcal{H} = 0$, once we embed the gauge connection into the torsional-spin connection. However, we can assume that our identification $\gamma = A = \tilde{\omega}$ is only to the lowest order in α', as we do not want to embed the gauge connection into the torsional-spin connection. This would mean, that there would be corrections to the three-form equation (2.47), implying that the sigma-model action would also receive corrections.[4] Observe, that all these corrections are

[4]Note that on the right hand side of (2.47) the ω and $\mathcal{H}$ are actually one-forms as in (2.46), whereas the left hand side $\mathcal{H}$ is a three-form. Since Ω_3 eventually makes a three-form, we hope that this will not confuse the readers.

JHEP04(2003)007

of order $(t^2/4\pi\alpha')^{-1}$, where t is the radius of our manifold. Therefore, they are small, if the radius is big. Thus we will assume

$$A = \tilde{\omega} + \mathcal{O}(\alpha'), \tag{2.49}$$

which leads to $d\mathcal{H} \neq 0$. In the following we shall use (2.47) to study the constraints on the size of our six-manifold. Details on this will be addressed in section 4.

However, there is still an ambiguity in defining the connection.[5] This appears explicitly, when we demand, that our sigma-model action (2.29) to be invariant under world-sheet supersymmetry transformations. If the supersymmetry transformation parameter is ϵ^p, then we require $\widehat{D}_+\epsilon^p = 0$, where $\widehat{D}_+$ is the same operator as in the second equation of (2.31), but now defined with respect to the connection $\widehat{\omega}$ instead of $\tilde{\omega}$, where

$$\widehat{\omega}_i^{ab} \equiv \omega_i^{ab} + \frac{1}{2}\mathcal{H}_i^{ab}. \tag{2.50}$$

In fact this connection is more relevant for deriving the torsional constraints of our model.[6] Therefore, this seems to leave us with the ambiguity in calculating the curvature tensor appearing in (2.6). However, observing (2.47) we see that we can, in fact, shift this ambiguity into a redefinition of the B field and keep the curvature tensor appearing in (2.6) unambiguous. Transforming B to

$$B \;\to\; B - \frac{\alpha'}{2}\,\omega \wedge \mathcal{H}, \tag{2.51}$$

where the one-forms ω and $\mathcal{H}$ are created from the corresponding three-forms by contracting with the vielbeins $e^{ai}e_a^j$, the curvature tensor is now defined with respect to ω. Therefore, the obstruction

$$\int \left[\mathrm{tr}\,(R\wedge R) - \frac{1}{30}\mathrm{Tr}\,(F\wedge F)\right] = 0, \tag{2.52}$$

is defined with respect to ω and is independent of the choice of connection. In passing, note that in (2.47) even though we have shifted the ambiguity into a redefinition of B, the three-form $\mathcal{H}$ still appears on both sides of (2.47).

Finally, observe that even for a non-zero $d\mathcal{H}$, i.e $A \neq \tilde{\omega}$, as long as we have the Killing spinor equation $\widehat{D}_+\epsilon^p = 0$, the two loop beta function (at least for the metric and to the lowest order in α') is trivial. As argued in [16], this may still be true to higher orders in α'. However, this argument relies on having a non-Kähler manifold, which is a *deformation* of the usual Calabi-Yau space and as such supports torsion, which is $\mathcal{O}(\alpha')$. Also as

[5] The Chern-Simons term in (2.47) arises from anomaly cancellation condition and there can, in principle, be *any* connection to cancel the anomalies. The difference, however, is that, by choosing another connection we have to introduce set of counterterms in the theory. Therefore we will stick to the case when the connection in (2.47) is (2.46) and elsewhere it is given by (2.50). This aspect have been discussed, for example, in [15]. We thank Chris Hull for correspondences on this issue.

[6] In [1] the connection used was in fact $\omega - \mathcal{H}$, which differs from (2.50) by a relative sign and also a relative factor of one half. Thus, the torsional equation of [1] has an extra minus sign and a missing factor of 2, as was also pointed out in [6]. However, since the torsion used there was $-\mathcal{T}$ this discrepancy doesn't alter any results.

argued in [17], the conformal invariance of these backgrounds may be spoiled by non-perturbative effects like world-sheet instantons. For our case, the torsion is of order 1 and the backgrounds receive $\mathcal{O}(\alpha')$ corrections. This case is different from [16] and even though we might have one-loop finiteness the fact that we demand a vanishing beta function for our case is a little subtle because of the size constraints of our six-manifold [18]. In particular, it can be shown that the corrections to the two loop beta function are suppressed, as long as the size of the manifold is sufficiently large. More details on this and whether we can have a large sized internal manifold will be addressed in section 4.

2.5 The anomaly relation in the heterotic theory

Until now we haven't taken the localized fluxes in (2.1) too seriously. In the $\mathcal{M}$-theory setup the localized part of the G-flux is of the form

$$\frac{G}{2\pi} = \sum_{i=1}^{4} F^i \left(z^1, z^2, \bar{z}^1, \bar{z}^2\right) \wedge \Omega^i \left(z^3, z^4, \bar{z}^3, \bar{z}^4\right), \tag{2.53}$$

where the supersymmetry constraints on G imply the primitivity condition $g^{a\bar{b}} F^i_{a\bar{b}} = 0$ on the gauge fields. In (2.53) we are summing over four fixed points, where at every fixed point there are four singularities. The two forms $\Omega^i_{a\bar{b}}$ are defined on the first $T^4/\mathcal{I}_4$ in $\mathcal{M}$-theory, while $F^i_{a\bar{b}}$ are located on the second $T^4/\mathcal{I}_4$, around which the $D7$ branes and $O7$ planes wrap. As discussed in [6], these localized fluxes are responsible for the gauge fields on the seven-branes on the type-IIB side. We are ignoring the constant fluxes for the time being.

The localized forms Ω^i are given by the 16 non-trivial $(1,1)$ forms on $T^4/\mathcal{I}_4$. Let us concentrate to a region near one of the singularities. This space will look like a Taub-NUT (TN) space and the (1,1) form is the unique (1,1) form of the TN space. This space has one radial coordinate r and three angular coordinates (θ, φ, ψ) with identification

$$(r, \theta, \varphi, \psi) \equiv (r, \pi - \theta, \pi - \varphi, -\psi). \tag{2.54}$$

The metric on this space is very well known. There is one singular point at $r = 0$, where the circle parametrized by ψ shrinks to zero. This circle is non-trivially fibred over the base (r, θ, φ). The anti-self-dual harmonic form on this space can be written in terms of real coordinates as [19]

$$\Omega = \mathrm{f}_o\, e^{-\mathrm{f}_1(r)} (d\sigma - \mathrm{f}_2(r)\ dr \wedge \sigma), \tag{2.55}$$

where, as discussed in [19], at large distances, i.e $r \to \infty$, $\mathrm{f}_1(r) = \frac{1}{2}r$, $\mathrm{f}_2(r) = \frac{1}{2}$, while f_o is a constant. Hence Ω is normalizable. The one-from σ is defined as

$$\sigma = -\sin\psi\, d\theta + \cos\psi \sin\theta d\varphi. \tag{2.56}$$

Inserting this in (2.53) we immediately face a problem. From the form of (2.55) we see, that Ω depends on $z^i, \bar{z}^j$, $i, j = 3, 4$. But to go to the type-I theory (or heterotic) we require it to be independent of these coordinates! Therefore, we have to study this space locally, near $r = 0$ and ignore all other fixed points. This in particular means, that we can only trace the behavior of *one* gauge field in the heterotic theory.

Near $r \to 0$, Ω is well defined and the behavior can be written explicitly as [19]

$$\Omega = \mathrm{f}_o \left(d\sigma_o - \frac{2}{\pi}\, dr \wedge \sigma_o \right) . \tag{2.57}$$

The above form is near the so-called bolt singularity. The one form σ_o is now different and is given by

$$\sigma_o = d\psi_o + \cos\theta_o \, d\varphi_o \, , \tag{2.58}$$

where $(\psi_o, \theta_o, \varphi_o)$ are the appropriate coordinates near the bolt [19]. The above is basically a constant form, if we ignore the angular dependences. Parametrizing appropriately, we can in fact write (2.57) as a constant $(1,1)$ form

$$\Omega = a\, dz^3 \wedge d\bar{z}^4 + b\, d\bar{z}^3 \wedge dz^4 \, , \tag{2.59}$$

where a, b are constants, that can be determined from (2.57). Plugging this in (2.53), and using the fact that $dz^4 = dx^{10} + \mathrm{i}dx^{11}$, we can determine the three form fields $H \equiv H_{NS}$ and $H' = H_{RR}$ in the type IIB theory as

$$\begin{aligned} H \, &= a\, F_{1\bar{2}} dz^1 \wedge d\bar{z}^2 \wedge dz^3 + b\, F_{\bar{1}2} d\bar{z}^1 \wedge dz^2 \wedge d\bar{z}^3 \, , \\ H' \, &= -ia\, F_{1\bar{2}} dz^1 \wedge d\bar{z}^2 \wedge dz^3 + ib\, F_{\bar{1}2} d\bar{z}^1 \wedge dz^2 \wedge d\bar{z}^3 \, , \end{aligned} \tag{2.60}$$

where we have ignored the contribution from the constant fluxes. One can view the above as if we had taken a large sized manifold, where the flux densities (2.10) are essentially zero. The resulting type-IIB three-form potential is[7]

$$G_3 = -2ia\, F_{1\bar{2}} \; dz^1 \wedge d\bar{z}^2 \wedge dz^3 \, . \tag{2.61}$$

Now as discussed in [6], to go to the type-I theory we have to make two T-dualities along the toroidal directions. Even with the choice of extra localized fields (2.60), the only non-vanishing components of $\mathcal{H}$ have one leg along the z_3 or $\bar{z}_3$ direction. Indeed, it is easy to verify that

$$\mathcal{B}_{mn} = 6B_{[8m}B'_{n9]} = 0 \, , \qquad \text{for} \qquad m, n = 4, \ldots, 7 \, . \tag{2.62}$$

The equation of motion of the six-form potential $C^{(6)}$ of the type-I theory will lead to the Bianchi identity of the heterotic string as follows. First, observe that

$$H \wedge H' = 2ab\, F \wedge F \wedge dx^8 \wedge dx^9 \, , \tag{2.63}$$

where we have defined F as a $(1,1)$ form, such that $g^{a\bar{b}}F_{a\bar{b}} = 0$ and $F_{ab} = F_{\bar{a}\bar{b}} = 0$. These are basically the Donaldson-Uhlenbeck-Yau (DUY) equations for the gauge bundle. The explicit form of F will be discussed in [20]. For the time being we take F to be of the form

$$F = F_{1\bar{2}} \; dz^1 \wedge d\bar{z}^2 + F_{\bar{1}2} \, d\bar{z}^1 \wedge dz^2 \, . \tag{2.64}$$

[7] We are choosing the axion-dilaton $\varphi = i$, as this is still the solution, even in the presence of localized fluxes. This will be clear soon, when we show, that the torsional constraints are satisfied for this case too.

Inserting (2.63) into the type-IIB Chern-Simons coupling $D^+ \wedge H \wedge H'$, where D^+ is the four-form potential, results in an interaction on the world-volume of the D7-brane, since the part of $H \wedge H'$ containing the Yang-Mills fields is localized. After T-duality, the Chern-Simons interaction maps to the $C^{(6)} \wedge F \wedge F$ interaction of the type-I theory. On the other hand the non-localized piece of $H \wedge H'$ gives rise to the kinetic term for $C^{(6)}$ in the bulk. A quick way to see this would be as follows. Let us denote the coordinates of the fiber T^2 as 8,9; $K3$ as X, Y, Z, W and Minkowski as μ, ν, ρ, σ. Thus in our notation

$$\mathcal{M}^{10}_{[A,B,\ldots]} = K3_{[X,Y,Z,W]} \times \frac{T^2_{[8,9]}}{\mathbb{Z}_2} \times \mathbb{R}^4_{[\mu,\nu,\rho,\sigma]}\,. \tag{2.65}$$

All the coordinates are real and $A, B, C, \ldots$ run over full ten dimensional indices. Also the non zero components of B and B' are of the form (i, X) where $i = 8, 9$.

Consider the interaction $D^+ \wedge H \wedge H'$. After integrating by parts we can write one of the contributions as

$$\int_{\mathcal{M}^{10}} dD^+_{W\mu\nu\rho\sigma}\; B_{8X}\; H'_{9YZ}. \tag{2.66}$$

This interaction results from the kinetic term of $C^{(6)}$ in the type-I theory

$$\begin{aligned}\int \mathcal{H}^I \wedge *\mathcal{H}^I &= \int \mathcal{H}^I_{ABC}\mathcal{H}^I_{A'B'C'}\, \epsilon^{A'B'C'}{}_{DEFGHIJ} \\ &= \int g^I_{DD'} g^I_{EE'}\; \mathcal{H}^I_{ABC}\mathcal{H}^I_{A'B'C'}\, \epsilon^{A'B'C'D'E'}{}_{FGHIJ}\,,\end{aligned} \tag{2.67}$$

by using two T-dualities. We will use the fact, that the $\mathcal{H}^I$ field has one component along the T^2 direction (here I denotes type-I components). For definiteness, consider the component

$$\int g^I_{8X} g^I_{99}\; \mathcal{H}^I_{8YZ}\mathcal{H}^I_{X'Y'Z'}\, \epsilon^{89X'Y'Z'}{}_{Wabcd}\,. \tag{2.68}$$

We can use the T-duality rules given in [21] taking into account, that the ϵ symbol of type-I is not the same as the ϵ symbol of type-IIB. In particular, the volume of T^2 has been inverted, and there are additional terms involving B fields. But we can write

$$\epsilon^{(I)89X'Y'Z'}{}_{Wabcd} = g_{88}g_{99}\epsilon^{89X'Y'Z'}{}_{Wabcd} + \cdots \tag{2.69}$$

where $\cdots$ are the terms involving B, B'. Inserting (2.69) into (2.68) and using the T-duality rules and the fact that the five-form is not closed, we get

$$\int dD^+_{Wabcd} B_{8X}\; H'_{9YZ} + \cdots \tag{2.70}$$

modulo signs and numerical factors. This is what we wanted to show.

Taking the previous interactions into account, the equation of motion for $C^{(6)}$ after U-dualities will be

$$d\mathcal{H} = -\alpha' F \wedge F\,, \tag{2.71}$$

where $\mathcal{H} = *dC^{(6)}$ and we have normalized a, b appropriately. The α' factor appearing above comes from the seven-brane. For the full non-abelian case (2.71) will take the form $\mathrm{tr}(F \wedge F)$. In deriving (2.71) we have used the orthogonality property of the $(1,1)$ forms.[8]

Observe that in our calculation we couldn't reproduce the complete anomaly relation for the heterotic string. This is because we considered the supergravity approximation, where we can see the contributions from the gauge bundle but not the contributions from the spin bundle. To get the complete anomaly, we will lift our solution to $\mathcal{M}$-theory, where the gravitational part of the anomaly is well known. To do so, let us consider the following chain of dualities [22]

$$\mathcal{M}\,\mathrm{on}\frac{T^4}{\mathcal{I}_4} \;\rightarrow\; \mathrm{IIB\ on}\,\frac{T^2}{\Omega\cdot(-1)^{F_L}\cdot\mathcal{I}_2} \;\rightarrow\; \mathrm{IIAon}\,\frac{T^3}{\Omega\cdot(-1)^{F_L}\cdot\mathcal{I}_3}\,. \tag{2.72}$$

From the above chain we see, that if in the type-IIB theory we have sixteen $D7$-branes plus four $O7$-planes, then we get in the type-IIA theory sixteen $D6$-branes plus eight $O6$-planes. From the type-IIA point of view every $D6$ branes lifts up in $\mathcal{M}$-theory to a Taub-NUT space and every $O6$ plane lifts as an Atiyah-Hitchin space satisfying [22]

$$\int_{\mathrm{Taub-NUT}} p_1 = \int_{\mathrm{Atiyah-Hitchin}} p_1 = \frac{1}{24}\int_{T^4/\mathcal{I}_4} p_1\,, \tag{2.73}$$

where p_1 is the first Pontryagin class of the spin bundle. The above relation directly translates to the type-I case by doing a series of U-duality transformations. The gravitational terms are in fact related to $\int_{T^4/\mathcal{I}_4} X_8$ in such a way, that the heterotic anomaly equation is

$$d\mathcal{H} = \frac{\alpha'}{2}[p_1(R) - p_1(F)]\,, \tag{2.74}$$

where $R = d\gamma + \gamma\wedge\gamma$, $F = dA + A\wedge A$ as defined earlier. As previously emphasized in [6], the two terms in (2.74) have different origins in $\mathcal{M}$-theory.

[8] This is easy to work out. Here we simply sketch the details: let us denote the twenty $(1,1)$ forms on $T^4/\mathcal{I}_4$ as $\Omega^i_{(1,1)}, i = 1, \ldots, h^{1,1}$ (replacing the orbifold by a smooth $K3$, we have to interpret the singularities as the points where the fiber torus degenerates on the base. As it is known, there is no simple way to write the metric or the harmonic forms for $K3$, and therefore the discussion for smooth $K3$ will be more involved. In this paper we will stick to the orbifold limit only). The G-flux in $\mathcal{M}$-theory can be written as:

$$\frac{G}{2\pi} = \sum_{i=1}^{4} A_{ij}\ \Omega^i_{(1,1)}(z^k,\bar{z}^k)\wedge\Omega^j_{(1,1)}(z^l,\bar{z}^l) + \sum_{p=1}^{h^{1,1}-4} F^p(z^k,\bar{z}^k)\wedge\Omega^p_{(1,1)}(z^l,\bar{z}^l),$$

where A_{ij} are constant numbers, $k = 1,2$ and $l = 3,4$. In the above equation the first sum is over the constant (1,1) forms and the other sum is over 16 localized forms. We have ignored the other two (2,0) and (0,2) forms (which are also constant forms). Using now the orthogonality property:

$$\int_{T^4/\mathcal{I}_4} \Omega^p_{(1,1)}(z^l,\bar{z}^l)\wedge\Omega^q_{(1,1)}(z^l,\bar{z}^l) = 16\pi^2\delta^{pq},$$

we can confirm the terms on the branes and the bulk. In deriving this, we have ignored the mass parameter of the TN space. These details do not alter any of the generic discussion, that we give here. Restoring it will simply add one more (determinable) parameter in the theory. This issue has been discussed earlier in [6].

JHEP04(2003)007

Finally, we should address the important question of torsional constraints in the presence of (2.60). Due to the anti-symmetrization in (2.60), we can write the NS-NS B field as

$$B = aA_{\bar{2}}\ d\bar{z}^2 \wedge dz^3 + bA_2 dz^2 \wedge d\bar{z}^3 \,. \tag{2.75}$$

This in particular will affect the heterotic metric because under U-duality NS-NS B-fields contribute to the metric. If we *ignore* the constant fluxes and take only the localized fluxes into account, then the heterotic metric will become

$$g_{a\bar{b}} = \begin{pmatrix} \Delta^2 & 0 & 0 \\ 0 & \Delta^2 + A_2 A_{\bar{2}} & -\frac{A_2}{2} \\ 0 & -\frac{A_{\bar{2}}}{2} & \frac{1}{4} \end{pmatrix}, \tag{2.76}$$

where we have ignored an overall factor of 2 and also assumed that the type-IIB coupling is $g_B = 1$, as this is a consistent solution [6]. For our case it is easy to see, that all the torsional constraints are satisfied. The only non-trivial relation is

$$\mathcal{H}_{21\bar{2}} = 2\, g_{\bar{2}[1,2]} \,. \tag{2.77}$$

As shown earlier in (2.62), $\mathcal{H}_{21\bar{2}} = 0$ and therefore, the metric derivative implies the following linear equation for the warp factor

$$\frac{\partial \Delta^2}{\partial z^1} + A_2\ \partial_1 A_{\bar{2}} = 0 \,. \tag{2.78}$$

This is precisely the expected dependence because, from the type-IIB point of view, the five-form equation is

$$F_5 = \frac{1}{2}(B \wedge H' - B' \wedge H) + \text{sources}\,, \tag{2.79}$$

where B, H' are the contributions from the *constant* fluxes (which we are ignoring for the time being) and the *sources* contribute precisely $A \wedge F$. The above derivation shows, that the torsional constraints are satisfied to the lowest order in α'.

2.6 Enhanced gauge symmetry

In [6] it was argued, that the full non-abelian symmetry in our model can actually appear from wrapped membranes on two-cycles of the $\mathcal{M}$-theory four-fold. Here we shall provide some details generalizing earlier work of [23]. This construction relies basically on the duality chain (2.72). We are also viewing the orbifold $T^4/\mathcal{I}_4$ as a $T^2/\mathcal{I}_2$ fibration over every fixed point of $T^2/\mathcal{I}_2$ and a T^2 fibration elsewhere. The metric can be written as

$$ds^2 = \Delta^{\frac{2}{3}} \left(g_{3\bar{3}} |dz^3|^2 + g_{4\bar{4}} |dz^4|^2 \right), \tag{2.80}$$

where Δ is the warp factor. In this space it is easy to construct a two-sphere as a cylinder shrunk at two points. A membrane wrapped on this sphere will have a mass given by

$$m_{ij} = \alpha \int_{r_i}^{r_j} \Delta^{\frac{4}{3}} d\psi\, dr \,, \tag{2.81}$$

where ψ, r are defined in (2.54). We denote the two points at which the cylinder shrinks as r_i, r_j, and α takes into account the dimensionful parameters. The above relation is not the full story. As argued in [23], there are additional two cycles, which contribute states with masses:

$$n_{ij} = \beta \int_{r_i}^{-r_j} \Delta^{\frac{4}{3}} d\psi\, dr\,, \tag{2.82}$$

which, in the type-IIB theory will be interpreted as states appearing due to the orientifold reflection. To see what kind of algebra we generate from above, we have to study the *intersection matrix*. This is constructed by considering the possible number of points where two spheres can intersection For our case, when we concentrate near the region, where we have $T^2/\mathcal{I}_2$ as our fiber, the intersection matrix $\mathcal{I}$ is easily shown to be

$$\mathcal{I} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & -1 & 0 & 2 \end{pmatrix}. \tag{2.83}$$

This is the Cartan matrix for the D_4 algebra and therefore globally we have a D_4^4 algebra and the enhanced gauge symmetry is realized when

$$m_{ij} = n_{ij} = 0\,, \qquad \forall i, j \tag{2.84}$$

and from the symmetry of the system we also have $\alpha = \beta$. In the language of type-IIB theory this is the case, when the $D7$ branes lie on top of an $O7$ planes and therefore the masses of the strings are zero.

3. Topological properties of non-Kähler manifolds

In this section we will study topological properties of Non-Kähler manifolds. We shall discuss various algebraic geometric properties of these manifolds and give a generic construction of a large class of them which are compact and also complex along with their complete Betti numbers etc. The basic algebraic geometric techniques used here can be extracted from [24]. Some of the details presented in section 3.2 are also obtained by [9].

We begin with a brief description of some of the lesser known properties of these manifolds.

3.1 Basic features of non-Kähler manifolds

(a) **Failure of identities on Hodge numbers.** Kähler manifolds have a great deal of structure that general complex manifolds have not, although the Hodge numbers h^{pq} are still defined for general compact complex manifolds. Here are some properties of Hodge numbers for Kähler manifolds that do not, in general, hold for compact complex manifolds.

The Hodge numbers do not completely determine the Betti numbers,the topological numbers, which characterize a general (real or complex) manifold M. The p-th Betti number b_p is the dimension of the p-th De Rham cohomology group $H^P(M)$, which depends only on the topology of M. By the Hodge theorem b_p counts the number of harmonic p-forms ω_p on M

$$\Delta_d \omega_p = 0\,, \tag{3.1}$$

with $\Delta = *d*d + d*d*$, where d is the exterior derivative as usual.

For a complex manifold one introduces the Dolbeault cohomology (or $\bar{\partial}$-cohomology) $H^{p,q}_{\bar{\partial}}(M)$, whose dimension is the Hodge number h^{pq}. The number of harmonic (p,q)

$$\Delta_{\bar{\partial}} \omega_{p,q} = 0\,, \tag{3.2}$$

which is (in general) not a topological invariant and is determined in terms of the Hodge numbers. Here one defines the operator

$$\Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^\dagger + \bar{\partial}^\dagger\bar{\partial}\,. \tag{3.3}$$

Only in the case of Kähler manifolds do the laplacians agree $\Delta_d = 2\Delta_{\bar{\partial}}$ and the Betti numbers become sums of Hodge numbers. In the non-Kähler case we still have the inequality

$$\sum_{p+q=n} h^{pq} \geq b_n\,,$$

but equality does not hold in general. Also the Hodge numbers are not in general symmetric; one may not assume that $h^{pq} = h^{qp}$.

Here is another important property of Kähler manifolds that can fail for general compact complex manifolds. If N is a compact complex submanifold of M, then a Kähler metric on M restricts to one on N. If ω is the Kähler class, and N has complex dimension n, it follows that $\int_N \omega^n > 0$. If M is not Kähler, it is possible that there is no closed $2n$ form on M whose integral over N does not vanish.

(*b*) **$\bar{\partial}$-cohomology and the Frolicher spectral sequence.** For any complex manifold M and any holomorphic bundle E over M, $\bar{\partial}$ is well defined as an operator on sections of E and, more generally, $\{0,q\}$ forms with values in E. This gives a complex whose cohomology is called the $\bar{\partial}$-cohomology with values in E. If we specialize to the case where E is the bundle of holomorphic $\{p,0\}$ forms, then the h^{pq} is defined to be the rank of the $\bar{\partial}$- cohomology. This is the meaning of the Hodge numbers for a general complex manifold. What is true in the Kähler case but fails in the general case is that every $\bar{\partial}$-cohomology class has a representative cycle that is ∂-closed as well. Instead, ∂ operates on the $\bar{\partial}$-cohomology, generating a new complex, and only the cohomology of this complex contributes to the cohomology of M. This is the first stage of what is called the Frölicher spectral sequence. In principle, there could be even further cancellations, but nobody seems to know of a case where this actually takes place.

(*c*) **The Serre spectral sequence of a fibration.** For the sake of simplicity, we will work throughout in the context of real cohomology. Let B and F be topological spaces. Then the Künneth formula asserts that

$$H^m(B\times F) = \sum_{p+q=m} H^p(B)\otimes H^q(F)\,, \tag{3.4}$$

for each non-negative integer m or, more compactly, $H^*(B\times F) = H^*(B)\otimes H^*(F)$.

We recall that a fiber bundle over B with fibre F is a space E with a projection $\pi : E \to B$ that is locally, but in general not globally, the product of B with F. The definition of a fibration with fiber F is more technical, but somewhat less restrictive. In either case, there is a spectral sequence $\{E_r^{pq}, d_r\}$ with

$$E_2^{pq} = H^p(B) \otimes H^q(F)\,, \qquad d_r : E_r^{pq} \to E^{p+r\,q-r+1} \tag{3.5}$$

is a homomorphism with $d_r^2 = 0$, and, suppressing the superscripts, E_{r+1} is the cohomology of the complex $\{E_r, d_r\}$. This is called the Serre (or Leray-Serre) spectral sequence. Because d_r has negative degree in the second superscript, E_r^{pq} eventually stablizes for each pq, even if B and F are not finite dimensional. This limit is called E_∞^{pq} and $H^m(E)$ has the same rank as $\sum_{p+q=m} E_\infty^{pq}$.

In particular, $E_\infty^{p,0}$ can be identified with a quotient of $H^p(B)$, namely

$$\pi * (H^p(B)) \subseteq H^p(E)\,. \tag{3.6}$$

The kernel of π^* is generated by the images of all d_r, the "eventual coboundaries" in the spectral sequence. Similarly

$$E_\infty^{0q} = \bigcap_r \ker\,(d_r)\,, \tag{3.7}$$

otherwise known as the "permanent cocycles" can be identified with a subgroup of $H^q(F)$, namely $j^*(H^q(E))$, where j represents the inclusion of F in E as a fiber.

One sees that the Betti numbers of the product $B \times F$ provide upper bounds for those of E. Moreover, d_r always has total degree $+1$, so that the Euler number of E is the product of those of B and F whether or not the fibration is actually a product.

We recall that the Künneth formula can actually be interpreted as the tensor product of the two cohomology rings. This gives a ring structure to E_2. It turns out that E_r has a ring structure for each value of r, for which d_r is a derivation so that E_{r+1} inherits its ring structure from E_r. In practice, this greatly facilitates computations.

However, the ring structure of $H^*(E)$ need not be identical with that of E_∞. What is really the case is that there is a filtration of $H^*(E)$ with E_∞ as the associated graded ring. While a filtered real vector space is necessarily isomorphic (although not canonically so) to its associated graded space, this is not the case for real algebras. The point is that rank is the only invariant of finite dimensional real vector spaces, but the structure of real algebras is more complicated.

There is also a Serre homology spectral sequence, $\{E^r_{pq}, d^r\}$ for which E^r_{pq} and d^r are the real duals respectively of E_r^{pq} and d_r, and E^∞ is the associated graded group to a filtration of $H_*(E)$. What is particularly important for our purposes is that an element of $H_*(F)$ maps to zero in $H_*(E)$ (is "homologous to zero") if and only if it is an eventual boundary in the Serre homology sequence.

It should be noted that, throughout this section, we have suppressed a complication involving the action of the fundamental group of the base on the homology or cohomology of the fibre. However this action is trivial in the case of a principal fiber bundle with connected group, and this is the only case we will be considering.

(*d*) **The case of a torus bundle.** A special case of the Serre sequence, and the only one that will concern us in the sequel, is that in which F is a torus $T = S \times S'$, where S and S' are copies of S^1, each with a fixed orientation. Then in the Serre sequence E_2^{pq} is non-vanishing only for $q = 0$, $q = 1$ and $q = 2$. It is immediate *a priori* that the only non-trivial d_r are d_2, and d_3 , and $E_4 = E_\infty$. Moreover, it follows from the ring structure, that if d_2 vanishes identically, then so does d_3. If we write s and s' for integral generators of $H^1(S)$ and $H^1(S')$ respectively, we may identify

$$\begin{aligned} E_2^{p0} &\mapsto H^p(B) \\ E_2^{p1} &\mapsto s \otimes H^p(B) \oplus s' \otimes H^p(B) \\ E_2^{p2} &\mapsto s \otimes s' \otimes H^p(B)\,. \end{aligned} \tag{3.8}$$

We write c and c' respectively for $d_2(s)$ and $d_2(s')$ in $H^2(B)$, and will refer to (c, c') as the double Chern class of T with respect to the decomposition $S \times S'$. Then we have

$$d_2(s \otimes s') = s' \otimes c - s \otimes c'\,, \tag{3.9}$$

where there cannot be any cancellation, since the two summands are in different "sectors" of E_2^{p1s}, and it is a straightforward consequence that if either c or c' is distinct from zero in $H^2(B, R)$, any generator of $H_2(T)$ is a boundary in the Serre homology sequence and so is homologous to zero in the total space. From this, it follows that the integral of any closed two-form over any fiber vanishes. In particular, if the total space is a complex manifold and the fibers are submanifolds, it follows that the total space cannot be Kähler.

If B is a simply-connected four dimensional manifold, we can compute the Betti numbers of E very easily. Since the Euler characteristic $\chi(T) = 0$, it follows that $\chi(E) = 0$. Since E will also be an orientable manifold and therefore satisfy Poincaré duality, $b_1(E)$ and $b_2(E)$ will determine all the other Betti numbers of E. But we have

$$\begin{aligned} H^1(E) &= \ker\left(d_2^{0,1}\right) \subseteq H^1(T)\,, \qquad \text{and} \\ H^2(E) &= H^2(B)/d_2^{0,1}(H^1(T)) \end{aligned} \tag{3.10}$$

since $E_3^{01} = E_\infty^{01}$ and $E_3^{20} = E_\infty^{20}$ are the only non-vanishing components of E_∞ of total degree 1 and 2 respectively.

This allows us to compute the Betti numbers of E, $b_i(E) \equiv b_i$, as

$$(b_0, b_1, b_2, b_3, b_4, b_5, b_6) = 1, 0, b_2(B) - 2, 2b_2(B) - 2, b_2(B) - 2, 0, 1 \tag{3.11}$$

if c and c' are linearly independent, and

$$(b_0, b_1, b_2, b_3, b_4, b_5, b_6) = 1,\ 1, b_2(B) - 1, 2b_2(B) - 2, b_2(B) - 1, 1, 1 \tag{3.12}$$

if c and c' are linearly dependent but do not both vanish as real cohomology classes. If c and c' are both trivial as real cohomology classes, then $E_2 = E_\infty$ and E has the same Betti numbers as $B \times T$.

JHEP04(2003)007

Thus for the example studied in (2.7), since the tori are all square, the real and imaginary part of the curvatures will be linearly independent and therefore the Betti numbers will be given by (3.11), i.e.

$$\{b_i\} = 1,\, 0, 20, 42, 20, 0, 1\,. \tag{3.13}$$

It follows from the remarks following equation (3.9), that the manifold is non-Kähler and has Euler characteristics, $\chi = 0$.

3.2 Holomorphic torus bundles

The examples of non-Kähler complex manifolds we shall consider are generalizations of the Hopf surface, which is the quotient of $C^2 - \{(0,0)\}$ by the multiplicative group of scalars generated by 2. The Hopf surface is a torus bundle over CP^1 since $C^2 - \{(0,0)\}$ is fibred over CP^1 by $C^\times$, the multiplicative group of non-zero complex numbers. The action of the discrete doubling and halving group is fiberwise and the quotient fibers are tori. We shall generalize this in two stages.

(a) **Torus bundles from holomorphic line bundles.** Let M be any complex manifold and L a holomorphic line bundle over M. Let $L^\times$ be the complement of the zero-section in the total space of L. Then $C^\times$, the multiplicative group of non-zero complex numbers acts holomorphically on $L^\times$. We divide out by the cyclic subgroup generated by any non-unimodular complex number, κ, to obtain a complex manifold N that is fibred over M with toral fibers. This is a direct generalization of the Hopf surface, for which $M = CP^2$, L is the Hopf bundle and $\kappa = 2$.

(b) **More general torus bundles.** Since a torus is a complex manifold, one can consider the most general holomorphic principal (meaning that the group of the bundle is the torus, acting on itself by translation) torus bundles. The usual construction of a bundle from transition functions applies in this case. Let T be a torus, and let M be a complex manifold. A holomorphic torus bundle can be obtained from a covering of M by open sets, U_α together with holomorphic maps

$$\phi_{\alpha\beta} : U_\alpha \cap U_\beta :\to T\,, \tag{3.14}$$

satisfying the usual requirements, namely $\phi_{\alpha\alpha}$ is the identity and $\phi_{\alpha\beta} \circ \phi_{\beta\gamma} = \phi_{\alpha\gamma}$ wherever both sides are defined.

(c) **An example.** Let M be the product of two copies of CP^1. Let z_1 and z_2 be homogeneous coordinates on the first factor, and let w_1 and w_2 be homogeneous coordinates on the second factor. Let T be defined as C/Γ, where Γ is a lattice of rank 2. Let U_{ij} be the open subset of M defined by $z_i \neq 0;\;\; w_j \neq 0$.

Let α and β be a basis of Γ. Define $\exp_\alpha$ and $\exp_\beta$ as maps from C to $C^\times$ defined respectively by $\exp_\alpha(z) = \exp(2\pi i \frac{z}{\alpha})$ and $\exp_\beta(z) = \exp(2\pi i \frac{z}{\beta})$. Then the quotient map $C \to C/\Gamma = T$ factor through each of the maps $\exp_\alpha$ and $\exp_\beta$. We will write π_α and

π_β for the other factor in each case, so that the quotient map can be expressed both as $\pi_\alpha \circ \exp_\alpha$ and $\pi_\beta \circ \exp_\beta$. With this notation, we define

$$\phi_{ij\; i'j'}((z_1, z_2), (w_1, w_2)) = \pi_\alpha\left(\frac{z_i}{z_{i'}}\right)\pi_\beta\left(\frac{w_j}{w_{j'}}\right). \tag{3.15}$$

It is straightforward to see, that the transition functions so defined satisfy the required relations. The total space of the torus bundle in this example turns out to be $S^3 \times S^3$, so that we obtain a family of complex structures on $S^3 \times S^3$, parametrized by the moduli space of tori. However, this example generalizes easily in several significant directions.

1. With no change of notation whatsoever, M can be replaced by the product of two complex projective spaces of arbitrary dimension, yielding complex structures on the product of any two odd dimensional spheres.

2. The transition functions $\phi_{ij\; i'j'}$ can be generalized by

$$\phi_{ij\; i'j'}((z_1, z_2), (w_1, w_2)) = \pi_\alpha\left(\frac{z_i}{z_{i'}}\right)^m \pi_\beta\left(\frac{w_j}{w_{j'}}\right)^n \tag{3.16}$$

   for any integers m and n; somewhat more generally, we can instead relax the requirement that α and β form a basis of the lattice, taking them to be any lattice elements at all.

3. We can increase the number of projective factors, associating a lattice element to each factor and proceeding as above.

4. We can increase the dimension of the torus. In this case, we have a lattice of rank $2n$ in C^n. If α is the lattice element associated to a projective factor, then the factor of the transition functions associated to that projective factor takes its values in the portion of the torus covered by the complex span of α. This will, in general, be a noncompact, possibly even dense, subset of the torus, but the construction remains valid.

5. Holomorphic torus bundles, like bundles in any category, transform contravariantly with respect to holomorphic maps so that any holomorphic map from a complex manifold M to any product of projective spaces induces many torus bundles over M.

(d) **Characteristic classes of holomorphic torus bundles of one complex dimension over Kähler manifolds.** Let $T = C/\Gamma$ as in the previous section, and let M be a Kähler manifold. We write, as usual, $\mathcal{O}$ for the sheaf of germs of holomorphic complex valued functions of M. We write $\mathcal{O}_T$ for the sheaf of germs of holomorphic T-valued functions on M. We can identify Γ with the sheaf of locally constant functions from M to Γ. This gives the exact sequence of sheaves

$$0 \to \Gamma \to \mathcal{O} \to \mathcal{O}_T \to 0 \tag{3.17}$$

which in turn yields a long exact cohomology sequence in which the terms of interest are

$$\to H^1(M,\Gamma) \to H^1(M,\mathcal{O}) \to H^1(M,\mathcal{O}_T) \to H^2(M,\Gamma) \to H^2(M,\mathcal{O}) \to \tag{3.18}$$

where holomorphic principal T-bundles over M are parametrized by $H^1(M,\mathcal{O}_T)$. The map

$$H^2(M,\Gamma) \to H^2(M,\mathcal{O}) \tag{3.19}$$

factors through $H^2(M,C)$, and the composition

$$H^1(M,\mathcal{O}_T) \to H^2(M,\Gamma) \to H^2(M,C) \tag{3.20}$$

permits us to define what we will call the complex chern class $c_\Gamma \in H^2(M,C)$ of the holomorphic principal T bundle. c_Γ is a linear combination of integral cohomology classes with coefficients in Γ. Because Γ has rank 2 over the integers, c_Γ is a linear combination of at most two integral classes. The map $H^2(M,C) \to H^2(M,\mathcal{O})$ (locally constant cochains to locally exact cochains) corresponds to projection of a complex cohomology class on its $(0,2)$ component, so that exactness of (3.18) at the penultimate term implies that the $(0,2)$ component of c_Γ must vanish. This is a necessary and sufficient condition for a complex linear combination of two integral cohomology classes to be the complex Chern class of a holomorphic principal torus bundle.

(e) **Connections and curvature.** Let E be the total space of a principal T bundle over the Kähler manifold M, where $T = C/\Gamma$. We observe that T acts on E as a transformation group with M as orbit space. If z is an affine coordinate on the universal covering space of T then dz is well defined as a one-form along the fibers, but not as a one-form on E. A holomorphic connection is defined to be a T invariant $(1,0)$-form ω on E whose restriction to each fiber coincides with dz. The adjective "holomorphic" in this context refers to the fact ω has no $(0,1)$ component; ω cannot be chosen to be a holomorphic form unless the bundle is flat (i.e. the transition functions are locally constant). There is no single natural choice for ω, but existence can be established using a smooth partition of unity on M together with holomorphic one-forms induced by local trivializations.

Two different holomorphic connections differ by $\pi^*(\theta)$ where $\pi : E \to M$ is the projection map and θ is a $(1,0)$ form on M. This observation makes the space of holomorphic connections an affine space over the space of $(1,0)$-forms on M.

The next observation is that if ω is a holomorphic connection on a torus bundle, then $d\omega$ is a closed two-form that descends to M and whose de Rham cohomology class is independent of ω. We will call it R_ω the curvature form of the connection. Although R_ω is not, in general, homogeneous with respect to the Hodge decomposition, it clearly has no $(0,2)$ component.

We will call ω a preferred connection, if R_ω is d-harmonic. Note that since M is Kähler, this implies that R_ω is harmonic with respect to all three of d, ∂ and $\bar{\partial}$, and that the $(1,1)$- and $(2,0)$- components of R_ω are separately harmonic, and therefore closed. To see that a preferred connection exists, let ω be any holomorphic connection and let R_E be the harmonic representative of R_ω. Let θ be a one-form with $d\theta = R_E - R_\omega$. Then the

$(0,1)$-component of θ is $\bar{\partial}$-closed, therefore anti-holomorphic, and therefore closed since M is Kähler. Hence it makes no contribution to $d\theta$, and may be assumed to be zero. Then $\omega + \pi^*(\theta)$ is a preferred connection. Moreover, a preferred connection is unique up to a holomorphic one-form on M.

We note next that for any local section σ of the torus bundle, any point of the E lying over the support U of the section is uniquely expressible as $t\sigma(m)$. This identifies $\pi^{-1}(U)$ holomorphically with $U \times T$ and, with respect to this decomposition, any holomorphic connection ω, preferred or not, has the form $dz+\theta$, where θ is a $(1,0)$-form on U. Changing the holomorphic section has the effect of changing θ by a holomorphic summand. This both provides a local recognition principle for holomorphic connections and identifies the space of holomorphic connections on a holomorphic torus bundle with $H^0(M, A^{1,0}/\Omega^1)$, where $A^{1,0}$ denotes the sheaf of smooth $(1,0)$-forms and Ω^1 denotes the sheaf of holomorphic one-forms.

To obtain a global recognition principal for holomorphic connections, we need to identify the cohomology class on M represented by the curvature form R_E. Let us begin by choosing an integral basis $\{\alpha, \beta\}$ for Γ and making the observation that T as a topological group is the product $S^1_\alpha \times S^1_\beta$, where S^1_α and S^1_β are respectively the subgroups of T covered by real multiples of α and β. Then there are integral cohomology classes c_α and c_β for which $c_\Gamma = \alpha c_\alpha + \beta c_\beta$. Moreover (c_α, c_β) is the double Chern class of the underlying topological torus bundle with respect to the decomposition $T = S^1_\alpha \times S^1_\beta$. We may, incidentally, conclude that E is not a Kähler manifold unless $c_\Gamma = 0$.

We write $E_\beta = E/S^1_\alpha$ and $E_\alpha = E/S^1_\beta$. Then E_α and E_β are circle bundles over M and E can be recovered as the fiber product of E_α and E_β. When we look at the torus bundle from this point of view, the components c_α and c_β of the double Chern class become respectively the Chern classes of the circle bundles E_α and E_β respectively.

Now let ω be a holomorphic connection and consider the forms

$$\begin{aligned} \omega_\beta &= \frac{1}{\bar{\alpha}\beta - \alpha\bar{\beta}}(\bar{\alpha}\omega - \alpha\bar{\omega})\,, \qquad \text{and} \\ \omega_\alpha &= -\frac{1}{\bar{\alpha}\beta - \alpha\bar{\beta}}(\bar{\beta}\omega - \beta\bar{\omega}) \end{aligned} \tag{3.21}$$

ω_α and ω_β are real, since, in each case, both numerator and denominator are imaginary. Moreover, $\omega = \alpha\omega_\alpha + \beta\omega_\beta$. ω_α and ω_β vanish along the orbits of S^1_β and S^1_α respectively and so descend respectively to one-forms on E_α and E_β. We can now see that $d\omega_\alpha = c_\alpha$ and $d\omega_\beta = c_\beta$. The usual factors of 2π and i are missing because ω_α and ω_β are real and have period 1 instead of 2π or $2\pi i$ on the fibers of E_α and E_β respectively. The conclusion is that $d\omega$ represents $\alpha c_\alpha + \beta c_\beta = c_\Gamma$.

This observation gives a necessary and sufficient condition for a harmonic two-form R on M to be a curvature form for some torus bundle: R must represent a complex linear combination of two integral cohomology classes, and the $(0,2)$-projection of R must vanish.

(f) **Metric, torsion and holomorphic forms.** For any holomorphic connection ω, $\frac{1}{i}\omega\wedge\bar{\omega}$ is a real $(1,1)$-form corresponding to a positive semi-definite hermitean inner product on the tangent bundle of the total space that is definite along the fibers. Let π be the projection from E to M, and let J_M be a Kähler form on M. It follows that

$$J_E = \frac{f}{i}\omega \wedge \bar{\omega} + g\pi * (J_M) \tag{3.22}$$

is a real $(1,1)$-form corresponding to a positive definite hermitean metric on the total space for any identically positive functions f and g.

The torsion is simplest to compute if we take f and g to be constant; clearly variable f and g will create additional terms in the torsion. We choose both to be constant and set $f = 1$. Then since J_M is closed, the contributions to ∂J_E and $\bar{\partial} J_E$ will come only from the first term. We further assume for simplicity that ω is a preferred connection so that $\partial\bar{\partial}\omega = 0$ and $\partial\omega$ and $\bar{\partial}\omega$ are separately harmonic. Then

$$\begin{aligned}
\partial J_E &= \frac{1}{i}(\partial\omega \wedge \bar{\omega} - \omega \wedge \partial\bar{\omega}) \\
\bar{\partial} J_E &= \frac{1}{i}(\bar{\partial}\omega \wedge \bar{\omega} - \omega \wedge \bar{\partial}\bar{\omega}) \\
\partial\bar{\partial} J_E &= -\frac{1}{i}\bar{\partial}\omega \wedge \partial\bar{\omega}\,.
\end{aligned} \tag{3.23}$$

Finally, we observe that if M has complex dimension n and Ω is a holomorphic $(n,0)$-form on M, then $\omega \wedge \pi^*(\Omega)$ is a holomorphic $(n+1,0)$ form on E, and is independent of the choice of ω. This guarantees that if M has a nowhere vanishing top dimensional holomorphic form, then so does E.

(g) **The case in which M is a product of two tori: the metric of (2.7).** If M is a product of two tori, $H^{2,0}(M)$ and $H^{0,2}(M)$ are each one dimensional. It follows that any two linearly independent integral cohomology classes in $H^2(M,Z)$ admit a complex linear combination without a $(0,2)$-component, so that there is no lack of holomorphic torus bundles over any product of two tori, or even over a more general torus of two complex dimensions.

However the case, as in the metric of (2.7), where the curvature is a pure $(1,1)$-form, is considerably more restrictive. In that case, we need to find a class in $H^{1,1}(M)$ that is a complex linear combination of two integral cohomology classes. Moreover, we have

$$H^{1,1}(T_1 \times T_2) = H^{1,1}(T_1) \oplus H^{1,1}(T_2) \oplus H^{1,0}(T_1) \otimes H^{0,1}(T_2) \oplus H^{0,1}(T_1) \otimes H^{1,0}(T_2) \tag{3.24}$$

and the form of the metric in (2.7) tells us that the class we need is orthogonal to the first two summands. For a generic choice of T_1 and T_2, such a class will not exist.

We note that the metric of (2.7) implies the connection

$$\omega = dz^3 + 2i\ \bar{z}^2 dz^1 - (4+2i)\ \bar{z}^1 dz^2\,, \tag{3.25}$$

where dz^3 is the standard holomorphic one-form along the fiber. This expression assumes the origin of the coordinates (z^1, z^2) to be at one of the fixed points of an involution of $T_1 \times T_2$. Because of its form, ω descends to the quotient of $T_1 \times T_2$ by the involution and coincides with dz^3 along the blowup of the fixed point. Changing fixed points entails adding a semi-period to each coordinate. This changes the form of ω by a holomorphic one form that is a linear combination of $\bar{s}_1 dz^2$ and $\bar{s}_2 dz^1$ where s_i is a semi-period of T_i. Moreover there is a sign ambiguity in the semi-period that corresponds to the sign ambiguity in the coordinates, so that the new form of ω still descends to the quotient. It follows from the foregoing that ω satisfies the local condition to be a holomorphic connection on a holomorphic torus bundle over the desingularization of $(T_1 \times T_2)/Z_2$.

To check the global condition, we note that the curvature form for the torus bundle implied by the metric of (2.7) is

$$R = 2i\, dz^1 \wedge d\bar{z}^2 - (4+2i)\, dz^2 \wedge d\bar{z}^1 \tag{3.26}$$

apart from an irrelevant real factor of $1/v$. Let us investigate what choices for the three tori (T_1, T_2, and the fiber T) involved are consistent with this choice.

We will assume that T_i has lattice generators $1, \mu_i$. It will always be possible to rescale without invalidating the following analysis, provided the tori are not rescaled independently. We choose integral cohomology generators a_i and b_i on T_i on the understanding that $a_i b_i$ is a positive multiple of the orientation class. Then we have $dz_i = a_i + \mu_i b_i$ and the curvature form for the torus is given, in terms of the integral generators as

$$\begin{aligned} R &= A_1\ a_1 b_1 + A_2\ a_1 b_2 + A_3\ b_1 a_2 + A_4\ a_2 b_2 \qquad \text{where} \\ A_1 &= 4+4i\,, \qquad A_4 = 2i\mu_1\bar{\mu_2} + (4+2i)\bar{\mu_1}\mu_2 \\ A_2 &= 2i\bar{\mu_2} + (4+2i)\mu_1\,, \qquad A_3 = 2i\mu_1 + (4+2i)\bar{\mu_2}\,. \end{aligned} \tag{3.27}$$

It is immediate by inspection that R will certainly be a complex linear combination of two integral classes if both μ_1 and μ_2 are gaussian integers (complex numbers with integral real and imaginary parts). In that case, T the torus of the fiber can be taken to have the gaussian integers as its lattice. This is the standard "square" torus.

4. Background superpotential

In the following we will be deriving the form of the superpotential, that is induced by the non-vanishing $\mathcal{H}$-flux in the heterotic theory compactified on a manifold with torsion. This superpotential gives rise to a potential for the moduli fields of the internal manifold. In the supergravity approximation we will basically follow a similar approach as in [25], for compactifications of $\mathcal{M}$-theory on Calabi-Yau four-folds. Thus, instead of extracting the superpotential from the potential for the scalars, we will be considering the dimensional reduction of a term quadratic in the gravitino appearing in the ten-dimensional action of $\mathcal{N} = 1$ supergravity coupled to Super-Yang Mills theory. In four dimensions the term quadratic in the gravitino has a coefficient proportional to the superpotential (see for example equation (25.24) of [26]), which makes the calculation easy. This is in contrast to the scalar potential, which is quadratic in the superpotential and also contains the derivatives $D_\alpha W$. Equivalently, we could have considered the dimensional reduction of the gravitino supersymmetry transformation to obtain the form of the superpotential, as has been done in [27] for the case of compactifications of the heterotic string on a Calabi-Yau three-fold.

But before we go into deriving the form of the superpotential , let us first discuss the existence of the holomorphic three-form for our background manifold (2.7).

4.1 Holomorphic three-form

The superpotential discussed in the type-IIB theory [11] is constructed from the holomorphic three-form Ω and G_3, made from the anti-symmetric tensors and the dilaton-axion. The three-form Ω is constructed from covariantly constant Weyl spinors η_- on the manifold and can be written as

$$\Omega = \eta_-^\top \Gamma_{123} \eta_- \, dz^1 \wedge dz^2 \wedge dz^3 \,, \tag{4.1}$$

where Γ_{123} is the anti-symmetrized product of three gamma matrices on the internal manifold as usual. The superpotential $\int G_3 \wedge \Omega$ survives on $T^4/\mathcal{I}_4 \times T^2/\mathbb{Z}_2$ and, as discussed in detail in [6], minimizing this superpotential with respect to the complex structure and the dilaton-axion we get the background constraints.

Now to go to the type-I theory we have to make two T-dualities along the T^2 directions. Under a single T-duality along, say, direction x^a, the chiral fermions transform as

$$\eta_- \;\; \to \;\; \tilde{\eta}_- = \sqrt{g_{aa}^{-1}} \, \Gamma_{11} \, \Gamma_a \eta_- \,, \tag{4.2}$$

where Γ_{11} is the ten-dimensional chirality operator. The holomorphic three-form in the type I/heterotic theory is then given by

$$\Omega = \frac{1}{\Delta^2} \, \eta_-^\top \Gamma_{123} \eta_- dz^1 \wedge dz^2 \wedge dz^3 \,, \tag{4.3}$$

where Δ is the warp factor. Notice, that the coefficient is $e^{-2\phi}$ as shown in [6], as the dilaton ϕ in the heterotic theory is proportional to the warp factor. We have also taken into account, that the gamma matrices in the heterotic theory differ from the corresponding gamma matrices in the type-IIB theory by some powers of the warp factor. As discussed in [1] and [6], even in the presence of the dilaton in Ω we get

$$\partial_\phi \Omega = 0 = \partial_{g_{mn}} \Omega \,, \tag{4.4}$$

and the holomorphic three-form is a non-trivial function only of the complex structure τ_{ij}. For a specific choice of complex structure, $\tau_{ij} = i\delta_{ij}$, we can use (2.35) to show, that Ω satisfies $\bar{\partial}\Omega = 0$, i.e it is a $\bar{\partial}$ closed $(3,0)$ form with everywhere non-vanishing $\int \Omega \wedge \bar{\Omega} > 0$. Here $\bar{\Omega}$ is the complex conjugate of Ω. And since Ω has no zeroes, we cannot multiply it with a meromorphic form to get linearly independent forms. Thus Ω is unique. Existence of a unique Ω is also consistent with the Bianchi identity of the three-form $\mathcal{H}$. The norm of Ω is given in terms of warp factor as

$$\begin{aligned} ||\Omega|| &= \left(\Omega_{123} \bar{\Omega}_{\bar{1}\bar{2}\bar{3}} g^{1\bar{1}} g^{2\bar{2}} g^{3\bar{3}} \right)^{\frac{1}{2}} \\ &= \frac{1}{\Delta^4} \left(1 + \frac{5|z^1|^2 + |z^2|^2}{\Delta^2} \right)^{\frac{1}{2}} , \end{aligned} \tag{4.5}$$

where we have used the inverse of the metric (2.34) and the normalization $\eta_-^\dagger \eta_- = 1$ for the covariantly constant spinors. In fact, we can interpret the norm of the holomorphic three-form in terms of *torsion classes.* Recently it was argued in the context of string theory compactifications, that the torsion of an SU(3) structure falls into five different classes called $\mathcal{W}_i$, $i = 1, \ldots, 5$ [7] and [8]. In the classification done by [7] and [8], the $\mathcal{W}_i$ for our torsional background (2.7) is given by

$$\begin{aligned} [\mathcal{W}_3] &= \mathcal{T}_{ijk}\,, \qquad [\mathcal{W}_1] \oplus [\mathcal{W}_2] = 0 \\ [\mathcal{W}_5] &= -2[\mathcal{W}_4] = a\frac{d\Delta}{\Delta} + b\frac{d\tilde{\Delta}}{\tilde{\Delta}}\,, \end{aligned} \tag{4.6}$$

where $\mathcal{T}_{ijk}$ is the torsion, a, b are constants and $\tilde{\Delta}$ is the modified warp factor. If we define $f(|z|) = 5|z^1|^2 + |z^2|^2$, then for our case

$$a = -4\,, \qquad b = \frac{1}{2}\,, \qquad \tilde{\Delta} = 1 + \Delta^{-2} f(|z|)\,. \tag{4.7}$$

More detailed analysis of torsion classes can be extracted from [8]. In fact the vanishing of Nijenhuis tensor is related to the vanishing of 1,2 torsion classes.

4.2 Superpotential

In ten dimensions the low-energy effective action describing $\mathcal{N} = 1$ supergravity coupled to Super-Yang-Mills theory contains (to leading order in α') a quadratic term in the gravitino Ψ

$$\Delta S_{10} = \int d^{10}x \sqrt{-g}\, \left(\bar{\Psi}_M \Gamma^{MNPQR} \Psi_R\right) \mathcal{H}_{NPQ}\,. \tag{4.8}$$

Here Γ^{MNPQR} is the anti-symmetrized product of ten-dimensional gamma matrices and $\mathcal{H}$ is the heterotic three-form, which in the supergravity approximation is given by $\mathcal{H} = dB - \alpha'\Omega_3(A)$, where A is the one-form potential of the non-abelian two-form and Ω_3 describes the Chern Simons form as usual.

We would like to compactify this interaction on the six-dimensional non-Kähler manifolds, which were described in the previous sections. In order to derive the form of the four-dimensional superpotential, the ten-dimensional Majorana-Weyl gravitino Ψ_μ is decomposed according to

$$\Psi_\mu = \psi_\mu \otimes \eta_- + \text{c.c.} + \ldots \tag{4.9}$$

where ψ_μ is the four-dimensional gravitino and η_- is the covariantly constant Weyl spinor of the internal manifold. The dots contain terms, that are needed in order to diagonalize the kinetic terms of the gravitino on the external and internal spaces.

Inserting (4.9) into (4.8) and decomposing the ten-dimensional gamma matrices, it is easy to see, that there appears a term in the four-dimensional effective action of the form

$$\Delta S_4 = \int d^4x \sqrt{-g}\, \left(\bar{\psi}_\mu \gamma^{\mu\nu} \psi_\nu\right) \cdot \int \mathcal{H} \wedge \Omega\,, \tag{4.10}$$

where Ω is the holomorphic $(3,0)$ form of the internal manifold and $\mathcal{H}$ describes the internal components of the flux. Therefore, in the supergravity approximation the superpotential takes the form

$$W = \int \mathcal{H} \wedge \Omega \,, \tag{4.11}$$

where we are integrating over the six-dimensional internal manifold. To this order the superpotential takes a similar form as the one derived in [27] for compactifications of the heterotic string on a Calabi-Yau three-fold. However, this superpotential is not the complete result for manifolds with torsion, as there are higher order contributions to the superpotential, which cannot be seen in the previous supergravity approximation.

One way to proceed to obtain the complete result is by using T-duality arguments, as follows. First, dB is basically related to the T-dual of the corresponding R-R form H' in the type-IIB theory (we are assuming a vanishing axion field). This is clearly given in [28], where they also argue that the NS-NS three-form of the type-IIB theory contributes to the spin-connection ω. Such contributions cannot be seen in the supergravity approximation previously done. Therefore, besides the obvious contribution from dB, T-duality rules can be used to show, that the superpotential receives further contributions from the curvature part living on the type IIB branes. In fact, a single type-IIB $O7$-plane and $D7$-brane, will give, after T-duality a contribution to the type-I superpotential of the form

$$W_1 = -e^{-\phi^I} \ \omega \wedge d\omega \,, \tag{4.12}$$

where ϕ^I is the type-I dilaton. However, as we discussed earlier, this is not the complete result. Due to the large number of branes and planes (and non-perturbative effects) the contribution from the curvature part is a little more involved, than the expression, that we get from a single system. The complete result can be written as[9]

$$W_2 = -e^{-\phi^I} \left(\omega \wedge d\omega + \frac{2}{3} \omega \wedge \omega \wedge \omega \right), \tag{4.13}$$

which in fact using our earlier notation is $\Omega_3(\omega)$, the Chern- Simons term. At this point we should take into account that our background has torsion. This will modify ω to

$$\omega_\mu^{ab} \ \rightarrow \ \tilde{\omega}_\mu^{ab} = \omega_\mu^{ab} - \mathcal{T}_\mu^{ab} \,, \tag{4.14}$$

where $\mathcal{T}$ is the torsion. Therefore, if we now take the gauge fields also into account and make an S-duality to go to the heterotic theory, we can identify the form of superpotential as

$$W = \int \mathcal{H} \wedge \Omega \,, \tag{4.15}$$

where $\mathcal{H}$ is not only given by dB but also contains the Yang-Mills and gravitational Chern-Simons terms, as discussed earlier in (2.47). We have also identified the torsion with the $\mathcal{H}$ field as in (2.45), and therefore is real and positive definite. However we will soon argue that there is another choice of superpotential which is *complex* and is useful to study backgrounds with non-Kähler geometry. The dilaton factor appearing in the holomorphic three-form above can be easily seen to appear in the string frame.

[9]A way to see this would be to look at the gravitational couplings on branes and planes in the type-IIB theory [22]. These couplings are distributed in some specific way for the $D7$- and $O7$-planes. As shown in the second reference of [22], this distribution is consistent with the full non-perturbative corrections to the brane-plane background.

An alternative way to derive the superpotential for the heterotic string on a manifold with torsion is to start with the superpotential of the heterotic string compactified on a Calabi-Yau three-fold [27] or equivalently with the superpotential derived in the supergravity approximation (4.11). In this approximation the flux involves the complete Chern-Simons term for the gauge field $\mathcal{H} = dB - \alpha'\Omega_3(A)$. We can now proceed with a similar logic as before. The only gauge invariant anomaly free expression for the flux appearing in the superpotential, should involve the Cherns-Simons term involving the spin connection $\Omega_3(\omega)$, which comes from a higher order term in the heterotic theory and cannot be seen in the supergravity approximation. If we now take into account, that our background has torsion, then the spin connection gets replaced by the torsional-spin connection $\tilde{\omega}$ and we obtain the same result for the superpotential, as the one we derived taking the D7-branes and O7-branes from the type-IIB theory into account (4.15).

Comparing (4.15) and (2.47) we see, that $\mathcal{H}$ appears on *both* sides of (2.47). Therefore, we have to solve (2.47) for $\mathcal{H}$ at every order in α' and plug the result into (4.15). First, we observe, that $d\mathcal{H}$ contributes an $\mathcal{O}(\alpha'^2)$. Secondly, the Chern-Simons term $\Omega_3(\tilde{\omega})$ can contribute (to a particular order in α') terms linear, quadratic and cubic in $\mathcal{H}$. Therefore, to *first* order in α', lowest order in the vielbein e^a and to a *linear* order in $\mathcal{H}$ we recover from (2.47) the known relation

$$\mathcal{H} = dB + \alpha'[\Omega_3(\omega) - \Omega_3(A)] + \ldots \equiv f + \ldots \tag{4.16}$$

where the dotted terms involve, to a given order in α', the pullback $\mathcal{H}$ as a one-form to the right hand side of (2.47).

The next order is α' is more involved, because we have to solve a differential equation to determine the value of $\mathcal{H}$. We will discuss about this soon. But before we go into the details of moduli stabilization we should mention a puzzle.

(*a*) **A puzzle related to T-duality.** As we had mentioned in the introduction, an independent calculation of the heterotic superpotential on manifolds with torsion has been performed by Tripathy and Trivedi [10]. The superpotential derived here and the one presented in the second reference of [10] do not look identical at first sight. In our case we see, that since $\mathcal{H}$ in (4.16) is real we obtain a real superpotential after inserting $\mathcal{H}$ into our superpotential (4.15), whereas the potential obtained by Tripathy and Trivedi is complex. The origin of the complex superpotential in their approach can be traced back from the type-IIB (or $\mathcal{M}$-theory) set-up. From (2.4) we see, that the G_3 flux of the type IIB theory has a real and an imaginary part. The form of the superpotential of the heterotic theory can be calculated by performing two T-dualities on the type-IIB superpotential. The three-form G_3 has a NS-NS part and a R-R part. The H_{NS} part is responsible for the gravitational (and partly) the Yang-Mills Chern-Simons terms on the heterotic side. The H_{RR} part gives us the three-form dB contribution to the superpotential and also contributes to the Yang-Mills Chern-Simons terms. Therefore, T-duality rules will give us a complex three-form for the type-I/heterotic theory.

The resolution to this discrepancy comes from the fact, that the three-form $\mathcal{H}$ defined in (2.47) appears on both sides of this equation. Therefore, to obtain the exact form of $\mathcal{H}$

we have to solve for $\mathcal{H}$ in (2.47). The analysis done in (4.16) is only to lowest order in α' and linear in $\mathcal{H}$. In fact, the background can be complex because, as we will argue in the next section, taking all contributions in α' into account gives a cubic equation from (2.47), which in general has real *and* complex roots. *This however doesn't mean that the heterotic three-form, that satisfies torsional equation and appears in the susy relations, is a complex quantity. By heterotic three-form we will always mean the real root of the cubic equation. The other two roots, that are complex conjugates of each other, are constructed from the real three-form plus a complex twist.* For the simplest case (where we only take the dependence on the radial modulus into account) the superpotential will look like (for simplicity we set $\alpha' = 1$ here, while in the rest of the paper we will keep track of α')

$$W = \pm \int \left(f + i\, b\omega + \sum_{m,n\, \in\, \mathbb{Z}/2} i c_{mn} f^m t^n \right) \wedge \Omega\,, \tag{4.17}$$

where b is a function of t and t is the size parameter of the non-Kähler manifold. The constants c_{mn} depend on α'. We have already defined f in (4.16), which contains $dB+$ Chern-Simons terms. In the next few sections we will determine the above summation in detail.[10] In fact, we will give a precise method, by which the generic heterotic superpotential can be determined to all orders. The simplest form, that we presented above in (4.17) is enough, to see how the radial modulus, would get stabilized. The ω dependent term is responsible for *twisting* the torus fiber, and the explicit appearance of t in (4.17) will fix the radial modulus. We can also see, how various terms in (4.17) are related to the corresponding T-dual type-IIB picture. The H_{NS} three-form flux of the type-IIB theory is responsible for the spin-connection ω in (4.17) and the dB part of f comes from the T-dual of H_{RR} flux [28, 10]. These are exactly the terms appearing in the superpotential of the second reference in [10]. Of course, T-duality rules are approximate and therefore, it will be difficult to obtain the complete result for the superpotential derived in this paper by using T-duality rules applied to the type-IIB superpotential. Most of the earlier works have ignored the gauge fluxes and do not see the Chern-Simons part of the superpotential.

4.3 Potentials for the moduli

In this section we will argue, that many of the moduli for the heterotic compactifications are lifted by switching on $\mathcal{H}$ fluxes.[11] We will show this for the particular example constructed in [6]. The basic strategy of the argument is as follows.

[10] A few points to consider here (more details will appear in the next sections): in the above formula for the superpotential (4.17) the terms in the summation are arranged so that $f^m t^n$ is *dimensionally* the same as f. For example, (as we will see soon) $\sqrt{t^3/\alpha'}$ is dimensionally the same as f. We will explicitly determine the first few terms in (4.17) when $m = 0, n = \frac{3}{2}$ and $m = 2, n = -\frac{3}{2}$. The rest of the terms can be easily determined from our general analysis. The $\pm$ sign in front of (4.17) represents the two possible choices of $\mathcal{H}$ that we have.

[11] An alternative way to fix moduli are discussed in [29]. Here use have been made of asymmetric orientifolds or duality twists to fix the Kähler moduli. It will be interesting to find the connection between our approach and theirs.

1. We compactify the heterotic theory on a six manifold, which is given by $T^4/\mathcal{I}_4 \times T^2$ with a vanishing expectation value for the fluxes. The important hodge numbers for our manifold are: $h^{11} = 21 = h^{21}$. This will determine the Kähler moduli and the complex structure moduli respectively. The metric for our manifold is well known and is given in terms of flat coordinates.

2. Now we turn on the three-form fluxes. This will back-react on the geometry by "twisting" the fiber torus, so that it is now non-trivially fibred over the $T^4/\mathcal{I}_4$ base.[12] The line element dz^3 of the fiber will change as

$$dz^3 \;\rightarrow\; dz^3 + 2i\; \bar{z}^2 dz^1 - (4+2i)\; \bar{z}^1 dz^2, \tag{4.18}$$

where we are ignoring the irrelevant factor of $\frac{1}{v}$ appearing in (2.10). The base $T^4/\mathcal{I}_4$ will also be changed by the warp factor Δ. The supersymmetry will reduce from $\mathcal{N} = 2$ to $\mathcal{N} = 1$. As discussed in [11] and [28], the fact that G_3 is of type (2,1) guarantees, that we have *at least* an $\mathcal{N} = 1$ supersymmetry. However in writing the three-form flux in the type-IIB theory as in (2.2), we have ignored two other contributions. First, the $(2,0) \oplus (0,2)$ choice of the G-flux in $\mathcal{M}$-theory and second the localized fluxes. In the presence of any (or both) of these contributions we can preserve exactly an $\mathcal{N} = 1$ supersymmetry, as we would have expected.

3. The three-from flux will actually change from dB to the value derived in (4.16) including higher order corrections. The kinetic term for the heterotic three-form flux, $\int \mathcal{H} \wedge *\mathcal{H}$, will give a potential for some of the moduli because it depends, first of all, on the complex structure. This comes from the definition of the coordinates $dz^i = dx^i + \tau^{ij} dy^j$, where x^i, y^j are the real coordinates and τ^{ij} are the complex structure parameters and from the choice of the harmonic forms. Second, due to the presence of the hodge star the potential depends on the metric and third, as briefly mentioned in (4.17), on the size t of the manifold. Let us illustrate the procedure by an example.

(*a*) **A toroidal example.** The toroidal compactification of SO(32) heterotic string broken down to a suitable subgroup has a Narain moduli space $\mathcal{M}_1$ given by [30]

$$\mathcal{M}_1 = \frac{\mathrm{SO}(n, n+16)}{\mathrm{SO}(n) \times \mathrm{SO}(16+n)} \qquad \mathrm{mod} \qquad \Gamma\,, \tag{4.19}$$

where Γ is the T-duality group $\mathrm{SO}(n, n+16, \mathbb{Z})$. For a generic compactification of the heterotic string on a T^n, we can illustrate the procedure mentioned above by which the moduli pick up masses, when suitable fluxes are switched on. This has been discussed to some extent in [31].

[12] A somewhat indirect reasoning to see this is the following. We start with the type-IIB theory on $K3 \times T^2/\mathbb{Z}_2$ without any fluxes. T-dualizing twice we get to the type-I on $K3 \times T^2$. Now on the type-IIB side, after switching on H_{NS} and H_{RR} fluxes, the geometry gets warped in the way shown in [4, 6] with no other changes. T-dualizing this configuration twice we get the type-I theory on a non-Kähler manifold. *Alternatively* one could think, that we switch on B fluxes in the type-I set-up. To *preserve* some supersymmetry the background has to be necessarily twisted. This is the "back-reaction" of the three-form fluxes on the geometry. In fact, it will soon be clear, that the superpotential does incorporate this twisting via the three-form fluxes defined in terms of the B fields plus the twist as explained briefly in (4.17).

The toroidal compactification of the heterotic string to $10-n$ dimensions has a gravity multiplet and some vector multiplets, that take the form

$$(g_{\mu\nu}, B_{\mu\nu}, \phi, nA_\mu) \;\oplus\; (n+16)(A_\mu, n\phi)\,, \tag{4.20}$$

with a moduli space $\mathcal{M}$ of dimension $[\mathcal{M}] = 1 + n(n+16)$. Apart from the dilaton, the Kähler moduli and complex structure moduli contribute $\frac{1}{2}n(n+1)$, the anti-symmetric tensor contributes $\frac{1}{2}n(n-1)$ and the sixteen abelian vectors contribute $16n$. The action for the moduli fields is given by

$$\int d^{10-n}\mathrm{x}\sqrt{-g}e^{-\phi}\left[(\partial_\mu\phi)^2 + \frac{1}{8}\partial_\mu M\cdot\partial^\mu M\right], \tag{4.21}$$

where the matrix M is given in [30]. If we denote the scalars coming from the Kähler structure and complex structure moduli as σ_i, the ones coming from the B field moduli as b_i and the ones from the vectors as a_i, then one can easily identify

$$\frac{1}{2}\partial_\mu M\cdot\partial^\mu M = \sum_{i=1}^{n(n+1)/2}(\partial_\mu\sigma_i)^2 - 2\sum_{j=1}^{16n}(\partial_\mu a_j)^2 - \sum_{k=1}^{n(n-1)/2}(\partial_\mu b_k)^2 + \cdots\,. \tag{4.22}$$

Observe, that in (4.22) all the scalars are massless, as they should and that the supersymmetry is $\mathcal{N} = 16$.

Let us now switch on fluxes. These fluxes are generically internal and appear on both, the Yang-Mills sector as well as the tensor sector. These fluxes take the following concrete form

$$F^a_{mn} = \alpha^a_{mn}\,, \qquad \mathcal{H}_{mnp} = \beta_{mnp} + \cdots\,. \tag{4.23}$$

The α fluxes are not arbitrary, but determined in terms of the fluxes β, because of the anomaly relation $d\mathcal{H} + tr\ F^a \wedge F^a = 0$. The reader can extract more details on this from [31]. Also, since we are considering a flat torus, all the curvature polynomials vanish. In particular $\mathrm{tr}R\wedge R$ would be zero.

In the presence of fluxes the lagrangian takes the form [31]

$$\int d^{10-n}\mathrm{x}\sqrt{-g}e^{-\phi}\left[\frac{1}{4}\sum_{i=1}^{n(n+1)/2}(\partial_\mu\sigma_i)^2 - \frac{1}{2}\sum_{j=1}^{16n}(\nabla_\mu a_j)^2 - \frac{1}{4}\sum_{k=1}^{n(n-1)/2}(\nabla_\mu b_k)^2 - V(\sigma)\right]. \tag{4.24}$$

Here we make the following observations:

1. The scalars coming from the two form anti-symmetric tensor field and the gauge fields have become charged. Therefore, their kinetic terms are given by covariant derivatives, defined as

$$\begin{aligned}\nabla_\mu a_i &= \partial_\mu a_i - \alpha_i\cdot\mathcal{A}_\mu\,,\\ \nabla_\mu b_i &= \partial_\mu b_i - \beta_i\cdot\mathcal{A}_\mu + \cdots\,,\end{aligned} \tag{4.25}$$

   where $\mathcal{A}_\mu$ are the Kaluza-Klein gauge fields.

2. The scalars representing the Kähler structure and complex structure moduli are not charged, but a *potential* $V(\sigma)$ is developed from the kinetic term of the three-form fluxes $\mathcal{H} \wedge *\mathcal{H}$, as well as from the gauge fluxes $F^a \wedge *F^a$. The explicit form of the potential is given by

$$V = g^{mm'} g^{qq'} g^{pp'} \beta_{mqp}\beta_{m'q'p'} + \sum_{a=1}^{16n} g^{mm'} g^{qq'} \alpha^a_{mq}\alpha^a_{m'q'}\,, \tag{4.26}$$

where g^{mp} is related to the scalars σ_i describing the Kähler and complex structure moduli (as these are determined from the metric). In fact (4.26), actually fixes *all* the complex structure moduli and some of the Kähler structure moduli.

3. As we shall soon see, the above consequences are quite generic. Switching on tensor fluxes, would give charges to the scalars coming from the Kaluza-Klein reduction of the tensor fields and would give a potential to all the complex structure moduli and some of the Kähler moduli. In the toroidal case studied above, the fluxes will also reduce the supersymmetry to some smaller value and would convert the T^n to some *twisted* T^n. This twist can, of course, be explicitly determined, if there exists a type-IIB or $\mathcal{M}$-theory dual of the model following the procedure of [6]. In the absence of a type-IIB (or $\mathcal{M}$-theory) dual the procedure to determine the twist is complicated. For $n = 6$, the case is subtle because sometimes, even if there would exist a type-IIB dual, the existence of fluxes may become forbidden, if the corresponding four-fold in $\mathcal{M}$-theory has zero Euler-characteristics [5]. For other values of n, there are no known obstructions. However, if we demand no supersymmetry for our background, then again there would be no obstruction for any values of n.

4. Notice also, that the dilaton moduli remains unfixed by the fluxes. This is in *contrast* to the type-IIB case studied in [11, 6] and [28]. In the type-IIB examples the potential is determined from the kinetic term $G_3 \wedge *G_3$, where $G_3 = H' - \varphi H$. Therefore, the dilaton-axion appears explicitly in the potential. For the heterotic theory this is not the case. However, as we shall soon see, for the heterotic case the *radial* modulus can, in fact, be fixed at tree level, which was not possible for the type-IIB case. In the later case higher order α' corrections have to be taken into account, in order to generate a potential for the radial modulus [25].

5. To summarize, the complete action for the heterotic theory in $10-n$ dimensions with gauge group broken to $U(1)^{2n+16}$ at any generic point will be given by

$$\int d^{10-n}\mathrm{x}\sqrt{-g}e^{-\phi}\left[R + (\partial\phi)^2 - \frac{1}{2}\mathcal{H}^2\right] + \mathrm{S}_{\text{gauge}} + \mathrm{S}_{\text{moduli}}, \tag{4.27}$$

where $\mathrm{S}_{\text{gauge}}$ is the action for the $16 + 2n$ gauge fields and $\mathrm{S}_{\text{moduli}}$ is the action for the moduli fields.

(*b*) **Moduli stabilization on the $K3 \times T^2$ manifold.** When heterotic string is compactified on a K3 manifold, the low energy effective theory is $\mathcal{N} = 1$ supergravity (8 supercharges) coupled to tensor multiplets, hypermultiplets and vector multiplets as

$$\left(g_{\mu\nu}, B^+_{\mu\nu}, \psi^+_\mu\right) \oplus \left(B^-_{\mu\nu}, \psi^-, \phi\right) \oplus 20(4\phi, \chi^-) \oplus 16\left(A_\mu, \lambda^+\right), \tag{4.28}$$

where $\pm$ denotes the chirality of Weyl spinors in six dimensions (in the subsequent discussion we shall ignore the fermions), while for the anti-symmetric tensor field this symbol indicates, if the field is self-dual or anti-self-dual. The sixteen vector multiplets are the contributions of the Cartan subalgebra of the gauge group.[13] Compactifying further on a torus and keeping a generic choice of the $\mathrm{n_V}$ vector multiplets and $\mathrm{n_H}$ hypermultiplets in four dimensions, we get a massless spectrum with $\mathcal{N}=2$ supersymmetry (the spinors are Majorana)

$$(g_{\mu\nu}, A_\mu, 2\psi_\mu) \oplus (\mathrm{n_V}+1)(A_\mu, 2\phi, 2\lambda) \oplus \mathrm{n_H}(4\phi, \chi)\,. \tag{4.29}$$

The extra vector multiplet can be traced from the six dimensional perspective. This is called the vector-tensor multiplet having $(\phi, B_{\mu\nu}, A_\mu, 2\lambda)$, where ϕ is the dilaton [32]. The anti-symmetric tensor is the axion in four dimensions and therefore this multiplet can be identified to be an *abelian* vector multiplet. As discussed in [32], the off-shell structure of this multiplet differs from that of the vector multiplet by the presence of a central charge, which vanishes on-shell. Furthermore, the axion-dilaton has a continuous Peccei-Quinn symmetry, which helps to determine the tree-level prepotential for all heterotic vacua. The loop corrections are severely restricted accordingly. We shall denote the axion-dilaton by $\tau = a - ie^{-\phi}$. As before, the action for the moduli is given by

$$\int d^4\mathrm{x}\sqrt{-g}\left[\frac{\partial_\mu\tau\partial^\mu\bar\tau}{(\tau-\bar\tau)^2} + \frac{1}{4}\sum_{i=1}^{\mathrm{n_V}}\partial_\mu z^i\partial^\mu\bar z^i - \mathrm{h}_{ab}\partial_\mu\sigma^a\partial^\mu\sigma^b\right], \tag{4.30}$$

where z^i are the complex scalars in the vector multiplets and the scalars σ^a are the scalars in the hypermultiplets, with h_{ab} as the metric on the moduli space. In fact, for our case when D_4^4 is broken to $\mathrm{U}(1)^{16}$, we can easily check, that $\mathrm{n_V} = 16, a = 1, \ldots, 21$. Of course , not all the scalars in the hypermultiplet are related to the Kähler moduli and complex structure moduli of the six manifold $K3 \times T^2$, as 22 of these scalars come from the anti-symmetric tensor. The moduli fields of the vector multiplets are contained in the M, appearing in (4.21). However, since the instanton number is constrained to be 24, the numbers of vectors and hypers depends on this constraint. The moduli space of hypermultiplets $\mathcal{M}_{\mathrm{H}}$ is a submanifold spanned by the moduli of the K3 surface $\mathcal{M}_{K3}$ and the moduli space of vector multiplets $\mathcal{M}_{\mathrm{V}}$ which, along with τ, span a special Kähler manifold. These moduli spaces are given respectively by [33, 32]

$$\mathcal{M}_{K3} = \frac{\mathrm{SO}(4,20)}{\mathrm{SO}(4)\times\mathrm{SO}(20)}\,, \qquad \mathcal{M}_{\mathrm{V}} = \frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \otimes \frac{\mathrm{SO}(2,\mathrm{n_V})}{\mathrm{U}(1)\times\mathrm{SO}(\mathrm{n_V})}\,, \tag{4.31}$$

where the τ spans the moduli space $\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}$. More details on the moduli space structure are given in [31].

[13] For the $E_8\times E_8$ heterotic string on $K3$, if we set the gauge connection to the SU(2) spin connection, then the 10d Yang-Mills multiplet will contribute

$$[(133,1)+(1,248)](A_\mu,\lambda^+) \oplus [10(56,1)+45(1,1)](4\phi,\chi^-)\,,$$

to the massless spectrum in six dimensions, where the terms in the bracket are the representations of the unbroken gauge group $E_7\times E_8$. The ten-dimensional gravity multiplet will contribute in the same way as before.

Let us now switch on a three-form flux, which breaks the space-time supersymmetry to $\mathcal{N} = 1$, and would convert the six-dimensional manifold $K3 \times T^2$ to the non-Kähler complex manifold discussed in [6]. The three-form flux takes the form

$$\mathcal{H} = a\,\Omega + \sum_{i=1}^{h_{21}} b^i \chi_i + \text{c.c.}\,, \tag{4.32}$$

where Ω is the holomorphic (3,0) form and χ_i are the (2,1) forms of the internal manifold. Observe, that the background three-form flux, that we switched on ignores the Chern-Simons contribution, and therefore it is related to the dB part of $\mathcal{H}$. For the subsequent discussion this doesn't affect much. We have also kept the background gauge fluxes to be zero for simplicity. Switching on the above three-form flux, the story should be the same as discussed earlier. The action for the moduli field will become

$$\int d^4\mathrm{x}\sqrt{-g}\left[\sum_{i=1}^{\mathrm{n_V}+1} G_{ij}\partial_\mu Z^i \partial^\mu \bar{Z}^j - \sum_{a,b=1}^{22} \mathrm{h}_{ab}\nabla_\mu \sigma^a \nabla^\mu \sigma^b + \sum_{c,d=1}^{61} \mathrm{h}_{cd}\partial_\mu \sigma^c \partial^\mu \sigma^d - V(\sigma)\right], \tag{4.33}$$

where we have combined the scalars τ and z^i into Z^i and defined a metric G_{ij} accordingly. The components of the metric can be easily ascertained from (4.30). Let us summarize the situation.

1. Some of the scalars of the hypermultiplet have become charged in the presence of the background fluxes. In fact, these scalars are precisely obtained by the dimensional reduction of the anti-symmetric two-form. The covariant derivatives for these scalars are defined with respect to a_i, b_i as before.

2. The scalars from the vector multiplet and the vector-tensor multiplet will remain massless, as we are not switching on the gauge fluxes. In the presence of gauge fluxes these scalars would also be charged.

3. Some of the scalars in the hypermultiplet, which come from the Kähler and complex structure of the six-manifold, will develop a potential V from the kinetic term of the three-form flux, written in terms of complex coordinates as $\int \mathcal{H}\wedge *\bar{\mathcal{H}}$. From the above choice of the three-form (4.32), the potential can be written explicitly as [34, 35, 11]

$$V = \frac{i}{2\mathrm{Im}\ \tau\ (\int \Omega\wedge\bar{\Omega})^2}\left[\int \Omega\wedge\bar{\Omega}\int \mathcal{H}\wedge\bar{\Omega}\int \bar{\mathcal{H}}\wedge\Omega + \int \chi^i\wedge\bar{\chi}^j \int \mathcal{H}\wedge\chi_i \int \bar{\mathcal{H}}\wedge\bar{\chi}_j\right]. \tag{4.34}$$

   The reason why this would fix some of the scalars in the hypermultiplet is, because the complex structure appears implicitly in writing the harmonic forms Ω and χ, when we decompose the three-forms in terms of h_{21} harmonic forms. This implies, that all the complex structure moduli will get fixed in the process. Fixing the radius moduli is however subtle, as we shall discuss in the next section.

4. Observe, that even though we turn on the gauge fluxes, making the scalars in the gauge multiplet charged, we *cannot* give a potential to the axion-dilaton τ. Hence,

JHEP04(2003)007

in this setup the axion-dilaton remains unfixed. Also, as we mentioned briefly at the beginning of the section, the presence of the B field twists the fiber. To preserve supersymmetry this is a consistent operation. However, this effect of the twisting should also be apparent from the form of superpotential, that we have in the heterotic theory. In fact, there is an additive term to the superpotential, that is proportional to the twisting, which does the job. This additive term is directly correlated with the existence of a complex contribution to the superpotential. We will discuss about this issue, when we determine the complete superpotential for the heterotic theory in the next section. This additive term in the superpotential is responsible for fixing some of the Kähler structure moduli. Therefore, to summarize the total number of moduli, that we could fix are the 22 scalars in the hypermultiplets, that become charged and the 21 complex structure moduli. Some Kähler moduli and the overall radial modulus also get stabilized.

4.4 Radial modulus

In type-IIB compactifications with fluxes the condition for unbroken supersymmetry is, that the flux G_3 should be a (2,1) form and that the (3,0), (0,3) and (1,2) parts should vanish. This requirement allows us to fix all the complex structure moduli and in some special cases some of the Kähler moduli. The condition that G_3 is a (2,1) form can be recast in the form

$$J \wedge G_3 = 0 \,, \tag{4.35}$$

which means that G_3 is a primitive form. As discussed in [11], a rescaling the metric or the fundamental form as

$$J \;\rightarrow\; tJ \,,$$

leaves the primitivity condition invariant and therefore the radial modulus will remain a free parameter, at least at the tree level. Therefore, the superpotential determines all the complex structure moduli and some of the Kähler structure moduli but not the radial modulus.

Making two T-dualities and an S-duality we go to the heterotic picture. On the heterotic side many of the type-IIB moduli have a different interpretation. Therefore, it is not surprising, that some moduli that remain unfixed in the type-IIB theory might be fixed in the heterotic setup. In this section we will argue, that the radial modulus can actually be determined in heterotic compactifications with torsion. Below we will give two reasons, that support this claim.

First, in the infinite radius limit the internal manifold cannot support the non-vanishing fluxes anymore and this leads to a contradiction. Indeed, imagine that the radius of the internal manifold could become arbitrarily large. In this limit the constant fluxes tend to vanish. In fact, from the choice of flux densities in (2.10), we see that as

$$v \;\rightarrow\; \infty \,, \qquad A \;\rightarrow\; 0 \qquad \text{and} \qquad B \;\rightarrow\; 0 \,, \tag{4.36}$$

even though the total flux integral is still non-zero. Therefore, the contribution to the total flux has to come from its localized part. In the large radius limit the fixed points go to infinity, but the fluxes in (2.60) still remain non-zero. From the warp factor equation (2.15) we see, that in the limit (4.36) the warp factor tends to be a constant

$$\Delta \rightarrow \sqrt{c_o}\,, \tag{4.37}$$

implying that the manifold is just a product $T^4/\mathcal{I}_4 \times T^2$. In this limit the torus is no longer non-trivially fibred over the base. Using now the torsional constraints (2.78), this would imply $A \wedge F = 0$, which is a contradiction because we have just shown, that localized sources survive in the large radius limit.

A second reason of why the radial modulus should be stabilized uses the torsional equation, which relates the background three-form field $\mathcal{H}$ to the two-form J as

$$\mathcal{H} = i(\partial - \bar{\partial})J\,. \tag{4.38}$$

We would like to see, how the left hand side of this equation transforms under $J \to tJ$, i.e. we would like to study the behavior of $\mathcal{H}$ in (2.47) under this transformation. Defining

$$\tilde{\mathcal{H}} = \mathcal{H}e^{-2} \equiv \mathcal{H}_{ijk}e^{aj}e^{bk}\,,$$

one can easily show, that the Chern-Simons term related to the torsional-spin connection $\tilde{\omega}$ is given by

$$\Omega_3(\tilde{\omega}) = \Omega_3(\omega) + \frac{1}{4}\Omega_3(\tilde{\mathcal{H}}) - \frac{1}{2}\left(\omega \wedge \mathcal{R}_{\tilde{\mathcal{H}}} + \tilde{\mathcal{H}} \wedge \mathcal{R}_\omega\right), \tag{4.39}$$

where we define $\Omega_3(\tilde{\mathcal{H}})$ in somewhat similar way as $\Omega_3(A)$ or $\Omega_3(\omega)$:

$$\Omega_3(\tilde{\mathcal{H}}) = \tilde{\mathcal{H}} \wedge d\tilde{\mathcal{H}} - \frac{1}{3}\tilde{\mathcal{H}} \wedge \tilde{\mathcal{H}} \wedge \tilde{\mathcal{H}}\,.$$

The quantity $\mathcal{R}_{\tilde{\mathcal{H}}}$ is the curvature polynomial due to the torsion and is defined as

$$\mathcal{R}_{\tilde{\mathcal{H}}} = d\tilde{\mathcal{H}} - \frac{1}{3}\tilde{\mathcal{H}} \wedge \tilde{\mathcal{H}}\,,$$

whereas $\mathcal{R}_\omega$ differs from the usual curvature polynomial by $-\frac{1}{3}\omega \wedge \omega$. In fact, we can write (4.39) in a more compact form as

$$\Omega_3(\tilde{\omega}) = \left(\omega - \frac{1}{2}\tilde{\mathcal{H}}\right)\left(\mathcal{R}_\omega - \frac{1}{2}\mathcal{R}_{\tilde{\mathcal{H}}}\right), \tag{4.40}$$

with the curvature polynomials defined above.[14]

[14] In this form it is instructive to compare with the other choice of torsional-spin connection $\widehat{\omega}$

$$\Omega_3(\widehat{\omega}) = \left(\omega + \frac{1}{2}\tilde{\mathcal{H}}\right)\left(\mathcal{R}_\omega + \frac{1}{2}\mathcal{R}_{\tilde{\mathcal{H}}} + \frac{1}{3}\tilde{\mathcal{H}} \wedge \tilde{\mathcal{H}}\right),$$

which differs from (4.40) in relative signs and an additional term. As discussed before, for some purposes we need $\tilde{\omega}$ and for others $\widehat{\omega}$. In fact, for deriving the torsional equations we need the connection $\widehat{\omega}$, as this is more appropriate (see section 2.4). However, for the time being we take the connection $\tilde{\omega}$, as this appears in the anomaly relation. We shall point out the consequence of the $\pm$ ambiguity in the connection later on, but one immediate thing to notice appears from comparing (4.39) with (2.51). In fact, we see, that

From the above analysis it is easy to infer, what the background torsion is. If we concentrate only to the lowest order in α' and linear order in $\mathcal{H}$, the three-form background is given by

$$\mathcal{H} = dB\Big(1 - \frac{\alpha'}{2}\mathcal{R}_\omega e^{-2}\Big) + \alpha'[\Omega_3(\omega) - \Omega_3(A)] + \mathcal{O}(\alpha'^2)\,. \tag{4.41}$$

Comparing (4.38) and (4.41) it is easy to see, that $J \to tJ$ is no longer a symmetry. Indeed, the two sides of (4.41) transform as

$$\mathcal{H} \;\to\; t\mathcal{H} \qquad \text{and} \qquad \mathcal{R}_\omega e^{-2} \;\to\; t^{-1}\mathcal{R}_\omega e^{-2}\,.$$

Observe that in (4.41) the term involving $\mathcal{R}_\omega e^{-2}$ is rank zero. To the *same* order in $\mathcal{H}$ we have ignored $-\frac{\alpha'}{2}\omega \wedge d\tilde{\mathcal{H}}$, which could also contribute. But since $d\tilde{\mathcal{H}} \sim \mathcal{O}(\alpha')$ this term is irrelevant.

Since the terms in (4.41) scale differently under rescaling of the metric we conclude, that there must exist an upper limit for the size of the torsional manifold. To all orders in $\mathcal{H}$ and α' the equation, that we need to solve is

$$\mathcal{H} + \frac{\alpha'}{2}\left[\omega \wedge \mathcal{R}_{\tilde{\mathcal{H}}} + \tilde{\mathcal{H}} \wedge \mathcal{R}_\omega - \frac{1}{2}\tilde{\mathcal{H}} \wedge \mathcal{R}_{\tilde{\mathcal{H}}}\right] = dB + \alpha'[\Omega_3(\omega) - \Omega_3(A)]\,. \tag{4.42}$$

From the above two reasons we find, that the radial modulus can in fact be controlled in this setup. As a result, there should be a potential for this modulus. This potential would follow from the form of the lagrangian, after taking the non-vanishing fluxes into account. This lagrangian involves the conventional Einstein term coupled to the kinetic term for the flux. After expanding the Einstein term the action takes the form

$$\int \sqrt{\det g}\, g^{-1}\left[\partial\left(g^{-1}\partial g\right) + g^{-2}(\partial g)^2\right] + \int \sqrt{\det g}\, g^{-3}\mathcal{H}^2\,, \tag{4.43}$$

where we integrate over the compact non-Kähler six-manifold. In this form, the lagrangian will be *invariant* under $g \to tg$, if we also set $\mathcal{H} = dB$, implying that the radial modulus does not receive a potential. This is what usually happens. However, as we have seen in the previous sections $\mathcal{H}$ is no longer dB, but is given by (4.41). Does this give a potential to the radial modulus?

---

when we shift the ambiguity in the connection into a redefinition of the two-form B, the three-form could be written completely in terms of the gauge fields A and the affine-connection ω without recourse to any torsional-connection. Calling the shifted field strength of the B field in (2.51) as h_{ijk}, we can easily confirm

$$\mathcal{H} = h + \alpha'[\Omega_3(\omega) - \Omega_3(A)] + \text{covariantterms}\,.$$

These covariant terms can be determined directly from (4.39) and are globally defined, whereas the spin connection ω and gauge fields A have to be defined on patches. This clearly tells us, that for any given four-cycle $\mathcal{C}$, the constraint

$$\int_{\mathcal{C}} d\mathcal{H} = \alpha' \int_{\mathcal{C}} [d\Omega_3(\omega) - d\Omega_3(A)] = 0\,,$$

which leads to (2.52), is defined with respect to either of the connections ω, $\tilde{\omega}$ or $\widehat{\omega}$ and is therefore unambiguous.

JHEP04(2003)007

It turns out, that to the order in (4.41), this *fails* to give a potential for the radial modulus. To see this, let us denote the radius by t and write locally the metric components as $g_{\mu\nu} = t\ \eta_{\mu\nu}$, where μ, ν span the six compact directions. The first terms in (4.43) give us the kinetic term for t, i.e $\partial_\mu t\ \partial^\mu t$. The spin connection one-form ω scales as $t^{-1}\partial t$. Therefore, to lowest order in α' and linear order in $\mathcal{H}$, the $\mathcal{H}$ term does not give a potential for the radius. In order to get a non-vanishing potential, we will have to consider the exact equation for $\mathcal{H}$, given by (4.42).

The reason of why the above argument failed is, that to the approximation taken in (4.41) all terms besides $\mathcal{H}$ contain either ω or $\mathcal{R}_\omega$ and therefore involve derivatives of t. But we also have terms cubic in $\mathcal{H}$, which originate from the third term in the bracket on the left hand side of (4.42), so that this equation becomes

$$\mathcal{H} + \frac{\alpha'}{12}\tilde{\mathcal{H}} \wedge \tilde{\mathcal{H}} \wedge \tilde{\mathcal{H}} + \cdots = dB + \alpha'[\Omega_3(\omega) - \Omega_3(A)]\,, \qquad (4.44)$$

where $\ldots$ represent the ω dependent terms. We can write the above equation as a cubic equation for $\mathcal{H}$

$$\tilde{\mathcal{H}}^3 + a\ \mathcal{H} + b = 0\,. \qquad (4.45)$$

The variables a, b can be determined from (4.44). In general, the equation (4.45) is not a typical cubic equation, because $\mathcal{H}$ defined above are forms on a manifold and not real numbers.[15] Therefore, let us first take a simple situation, which can convert (4.45) into a cubic equation with real variables. For the present purpose, this will suffice to clarify the basic ideas involved here. We will soon generalize this to our six dimensional non-Kähler manifold.

(*a*) **A simple toy example.** Let us choose a background, that can convert equation (4.45) into a simple cubic equation in terms of functions and not forms. The simplest background will be

$$\mathcal{H}_{ijk} = h\mathcal{C}_{ijk}\,, \qquad e_{ai} = t^{1/2}e^o_{ai} \qquad \text{and} \qquad e^o \cdot e^o = \eta\,, \qquad (4.46)$$

where $\mathcal{C}_{ijk}$ is an antisymetric in six dimensions, e_{ai} are the vielbeins and $\eta \equiv \eta_{ij}$ is the flat metric. The size of the six manifold is given by t, as defined earlier. We will also assume, that the only non-zero component of $\mathcal{H}$ is $\mathcal{H}_{12\bar{3}} = h$. It is easy to see, that (4.45) becomes a simple cubic equation

$$h^3 + ph + q = 0 \qquad \text{with} \qquad p, q \in \mathbb{R}\,, \qquad (4.47)$$

[15]The precise value of the cubic wedge products of $\tilde{\mathcal{H}}$ can be shown to be equal to

$$\tilde{\mathcal{H}} \wedge \tilde{\mathcal{H}} \wedge \tilde{\mathcal{H}} = \tilde{\mathcal{H}}^{ab}_i\, \tilde{\mathcal{H}}^{cd}_j\, \tilde{\mathcal{H}}^{ef}_k\, \mathrm{Tr}(M^{ab}M^{cd}M^{ef})dx^i \wedge dx^j \wedge dx^k\,,$$

where the one form $\tilde{\mathcal{H}}^{ab}_i = -\tilde{\mathcal{H}}^{ba}_i = \mathcal{H}_{ijk}e^{ai}e^{bk}$ and M^{ab} are the tensors of the holonomy group of the manifold. As can be seen this doesn't vanish generically and therefore gives rise to a cubic equation as (4.45). The one form $\tilde{\mathcal{H}}^{ab}_i$ shares similar properties with the spin connection $\omega^{ab}_i = \omega_{ijk}e^{aj}e^{bk}$.

where p, q are given below. The solution to this equation is well known using Vieta's substitution.[16] Observe, that for simplicity we have ignored all ω dependent terms and $\mathcal{O}(\alpha'^2)$ terms appearing in (4.42). To leading order in α' the presence of ω would induce quadratic and linear terms in $\mathcal{H}$ in the previous equation. This does not change the analysis, that we perform here, because *any* generic cubic equation can always be brought to the form (4.45) with additive contributions to p, q. For the simplest case we can specify p, q as

$$p \sim \pm \frac{t^3}{\alpha'}, \qquad q \sim \mp \frac{t^3 f}{\alpha'}, \qquad \text{and} \qquad \langle dB + \alpha' \Omega_3(\omega) - \alpha' \Omega_3(A) \rangle_{abc} \equiv f \epsilon_{abc}, \tag{4.48}$$

where the sign ambiguity reflects the sign ambiguity in the torsional-spin connection, $\omega \mp \frac{1}{2}\mathcal{H}$, i.e the choice of either $\tilde{\omega}$ or $\widehat{\omega}$ in terms of earlier notations. The reader might be concerned by the fact, that since (4.44) involves *wedge* products, it will be difficult to get a simple cubic equation. But it is easy to see, that all the other factors due to wedging can be absorbed into the definition of p and q and therefore, we can express (4.44) as a cubic equation in h with p, q *proportional* to (4.48), as we are not very concerned about precise factors. Nevertheless, the exact factors appearing here can be worked out easily. In this section we take a simple example, where the background is just $\mathcal{H}_{12\bar{3}}$. This implies

$$\tilde{\mathcal{H}}_1^{ab} = \mathcal{H}_{1\alpha\beta} e^{a\alpha} e^{b\beta} = 2h\; \mathcal{C}_{12\bar{3}} e^{2[a} e^{b]\bar{3}} \equiv 2t^{-1}\; h\mathcal{C}_{12\bar{3}} e_{\rm o}^{2[a} e_{\rm o}^{b]\bar{3}} = t^{-1} h \alpha_1^{ab}, \tag{4.49}$$

where $\alpha_1^{ab} = -\alpha_1^{ba} = 2\mathcal{C}_{12\bar{3}}\; e_{\rm o}^{2[a} e_{\rm o}^{b]\bar{3}}$ and $e^{ai} = t^{-\frac{1}{2}} e_{\rm o}^{ai}$. We have also defined the anti-symmetrization between a, b as $[\cdots]$, as usual. The above formula can easily be generalized to the case, in which other components of $\mathcal{H}$ are turned on and not all $\mathcal{C}$ are equal. Some aspects of this will be dealt with in the next sub-section. We are also using complex coordinates, but it is easy to infer the corresponding case with real coordinates. Similarly to the above equation we can obtain the other components of $\tilde{\mathcal{H}}_i^{ab}$.

$$\tilde{\mathcal{H}}_2^{cd} = 2h\; \mathcal{C}_{2\bar{3}1} e^{\bar{3}[c} e^{d]1} = t^{-1} h \alpha_2^{cd} \qquad \text{and} \qquad \tilde{\mathcal{H}}_{\bar{3}}^{ef} = t^{-1} h \alpha_{\bar{3}}^{ef}. \tag{4.50}$$

Now we can determine the cubic wedge product between the $\mathcal{H}$'s, using the result of the earlier footnote. We again use M^{ab} as the tensors of the holonomy group of the manifold. The result will be

$$\left[\tilde{\mathcal{H}} \wedge \tilde{\mathcal{H}} \wedge \tilde{\mathcal{H}}\right]_{12\bar{3}} = t^{-3} h^3 \alpha_1^{ab} \alpha_2^{cd} \alpha_{\bar{3}}^{ef}\, {\rm Tr}\left(M^{ab} M^{cd} M^{ef}\right) = t^{-3} h^3 \mathcal{Q}, \tag{4.51}$$

where $\mathcal{Q}$ is an integer determined in terms of α_i and M in an obvious way. In general, $\mathcal{Q}$ is a non-zero integer and for the analysis done in this sub-section and the next one, we shall

[16] Define $\mathcal{H} = w - \frac{p}{3w}$ and substitute it in (4.45). The equation will become a quadratic equation in $w^3 \equiv \lambda$, given by $\lambda^2 + q\lambda - \frac{p^3}{27} = 0$, whose solutions are $w^3 = \frac{1}{2}\left(-q \pm \sqrt{q^2 + \frac{4p^3}{27}}\right)$. Calling these roots as $\delta_\pm^3$, w can be written as $w = \delta_\pm(1, \omega, \omega^2)$, where ω is the cube-root of unity. The solutions are therefore $\delta_+ + \delta_-, \delta_+\omega + \delta_-\omega^2$ and $\delta_+\omega^2 + \delta_-\omega$. This is Cardan's solution for a cubic equation. For a generic cubic equation of the form $h^3 + a_2 h^2 + a_1 h + a_0 = 0$, we can define $h = y - \frac{a_2}{3}$. In this form the equation looks in the same way as (4.47) with $p = a_1 - \frac{1}{3}a_2^2$ and $q = \frac{1}{3}(3a_0 - a_1 a_2) + \frac{2}{27}a_2^3$. Calling the roots as α_i, we get: $\sum \alpha_i = -a_2$, $\sum \alpha_i\alpha_j = a_1$, $\sum \alpha_i\alpha_j\alpha_k = -a_0$. Observe also, that since $\frac{p^3}{27} + \frac{q^2}{4} > 0$, we have one real root and pair of complex conjugate roots.

JHEP04(2003)007

normalize $\mathcal{Q}$ to 1 by defining variables appropriately. This will not alter any of our results because we only require the form of the moduli and not the precise factors. (Later we will consider the case when $\mathcal{Q}$ vanishes.) From here the set of equations (4.48) can be derived. To determine the solution for a cubic equation, we define, as usual, two variables A and B as

$$A = \left(-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)^{1/3} \qquad \text{and} \qquad B = \left(-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)^{1/3}, \tag{4.52}$$

where p, q have been introduced above. Observe, that if $p \to \infty,\ q \to \infty,\ q/p =$ constant, we recover the situation, where no torsion is switched on. This implies, that expanding in $1/p$ is a legitimate thing to do (we will give further justification later). In fact, for the limit mentioned above we get $h \sim -q/p$, which is a reasonable estimate in the absence of torsion, because this implies $\mathcal{H} = dB + \mathcal{O}(\alpha')$. Therefore, the solution of the cubic equation for $\mathcal{H}$ (4.45) will be typically

$$A+B\,, \qquad -\frac{A+B}{2} + i\frac{\sqrt{3}(A-B)}{2}\,, \qquad \text{and} \qquad -\frac{A+B}{2} - i\frac{\sqrt{3}(A-B)}{2}\,, \tag{4.53}$$

giving two complex and one real solution.

This by itself is interesting, because it implies, that we can actually have a complex three-form in the heterotic string theory. This means, that the superpotential written in terms of $\int \mathcal{H} \wedge \Omega$ can have a complex part and this is precisely, what we have been looking for, when we mentioned the apparent i-puzzle in an earlier section. To see, whether the present solution is related to the solution obtained by performing T-dualities from a type-IIB background, we have to study the set of solutions (4.53) carefully. Let us define a quantity

$$s \equiv \sqrt{1+z^2} - 1 \qquad \text{with} \qquad z = \sqrt{\frac{27q^2}{4p^3}}\,,$$

where z is a small quantity and therefore s can be expanded in powers of z. We can express A and B in terms of these variables as

$$A = a(1+s-z)^{1/3} \qquad \text{and} \qquad B = -a(1+s+z)^{1/3}\,, \tag{4.54}$$

where $a = \sqrt{p/3}$. Expanding the above relations in powers of $1/p$, we can determine the solutions of (4.53) order by order. The solutions we obtain are

$$\begin{aligned} A+B &= -\frac{2az}{3} + \frac{4asz}{9} - \frac{10az^3}{81} - \frac{10as^2z}{27} + \frac{80as^3z}{243} + \cdots\,, \\ A-B &= 2a + \frac{2as}{3} - \frac{2a(s^2+z^2)}{9} + \frac{10as(s^2+3z^2)}{81} + \cdots\,. \end{aligned} \tag{4.55}$$

What we need to know is, how the background three-form field $\mathcal{H}_{12\bar{3}} = h$ depends on the size t of the six-manifold. For the real solution we get

$$h = f - \frac{\alpha' f^3}{t^3} + \cdots\,. \tag{4.56}$$

Comparing with (4.48) we see, that to the lowest order in α' this is exactly, what we expect. Observe also that, for the usual case where is the radius is not fixed – due to Dine-Seiberg runaway problem [18] – the radius will tend to go to infinity and therefore $h \to f$. Therefore this three-form h is the *real* three-form of the heterotic theory that satisfies the torsional equations and appear in the anomaly and susy relations. The other two complex solutions are

$$h = -\frac{f}{2} \pm i\sqrt{\frac{t^3}{\alpha'}} + \cdots . \tag{4.57}$$

These solutions do not satisfy the torsional equations because they are complex, but are anomaly free and gauge invariant. The real part of them are proportional to the usual heterotic three-form and the complex part provide the necessary *twist* to change the topology of our space from $b_1 = 2$ to $b_1 = 0$ (this will be more apparent when we will also include the spin-connection ω later in the section). This is a non-trivial constraint because by switching on a supergravity three-form we cannot change the topology of any space. Therefore we need some twist. This is precisely provided by our choice of the complex three-form! Thus, for the connection, that is relevant for the physics of non-Kähler manifolds, the background three-form should be real and complex. As seen from the analysis of [28], T-duality rules actually chooses the complex three-form.

It now remains to see, what the potential for the radial modulus in our toy model is. We shall choose the complex h in (4.57). It is easy to check, that the resulting potential takes the form

$$V(t) = \sqrt{g}\, g^{1\bar{1}} g^{2\bar{2}} g^{3\bar{3}}\ \mathcal{H}_{12\bar{3}}\ \mathcal{H}_{\bar{1}\bar{2}3} = \frac{t^3}{\alpha'} + \mathcal{O}(t^{-n})\,, \tag{4.58}$$

where the metric components appearing above have been scaled by t, i.e. $g \to tg$. It can also be checked, that the real parts of (4.57) always contribute a potential of order $\mathcal{O}(t^{-n})$, where n is an even integer. In fact, the only positive power of t comes from the second term of (4.57). Therefore, we can now express the action for the radial modulus as

$$\mathcal{L} = \partial_\mu t\ \partial^\mu t + \frac{t^3}{\alpha'} + \cdots , \tag{4.59}$$

implying, that the size of the six-manifold is determined by the scale α', appearing in the anomaly relation. We will give an estimate for the size of the six-manifold soon. But first we need to see, whether we can extend the above calculations to the realistic case of non-Kähler manifolds.

(*b*) **Extension to non-Kähler manifolds.** From the above analysis we concluded, that the size of the six-manifold can indeed be stabilized by a potential generated by the complex three-form flux. However, the model discussed above is not realistic, since we have chosen very simple fluxes. In this section we will consider the extension of the above analysis to non-Kähler six-dimensional manifolds.

The first difficulty we find, when dealing with the background (2.7) is, that $\mathcal{H}$ has many components. Thus, when we consider the equation (4.44) we have to be careful with the wedge products involved. As a consequence $\tilde{\mathcal{H}}$ will also be more complicated. In the toy example $\tilde{\mathcal{H}}$ was simply proportional to $t^{-1}h$, because only one component $\mathcal{H} = \mathcal{H}_{12\bar{3}}$ was non-vanishing.

From (2.7) we can see that the possible components of $\mathcal{H}$ are

$$\mathcal{H}_{\bar{1}2\bar{3}} \equiv h_1 \mathcal{C}_{\bar{1}2\bar{3}} \qquad \text{and} \qquad \mathcal{H}_{1\bar{2}\bar{3}} \equiv h_2 \mathcal{C}_{1\bar{2}\bar{3}}\,, \tag{4.60}$$

and their complex conjugates, which we denote as h_3 and h_4 respectively. In the analysis below we shall ignore the spin connection dependent terms, as they do not change the form of the cubic equation. The one-form $\mathcal{H}_i^{ab}$, for example, can be shown to be proportional to

$$\begin{aligned}
\mathcal{H}_{\bar{1}} &\equiv \mathcal{H}_{\bar{1}}^{ab} = \mathcal{H}_{\bar{1}jk}\, e^{ja}\, e^{kb} = t^{-1}\left(\alpha_{\bar{1}}^{ab}\, h_1 + \beta_{\bar{1}}^{ab}\, h_4\right),\\
\mathcal{H}_2 &= t^{-1}\left(\alpha_2^{ab}\, h_1 + \beta_2^{ab}\, h_4\right),\\
\mathcal{H}_{\bar{3}} &= t^{-1}\left(\alpha_{\bar{3}}^{ab}\, h_1 + \beta_{\bar{3}}^{ab}\, h_2\right),
\end{aligned} \tag{4.61}$$

where $\alpha_i^{ab}, \beta_i^{ab}$ are constants, whose exact form can be determined from the vielbeins. Due to (4.61) the cubic equation will be far more complicated, than the one discussed before. For simplicity we shall assume all components of $dB + \mathcal{O}(\alpha')$ to be proportional to a function f, as before. We will show later, that this simplification does not substantially affect the results. We will also concentrate mainly on the lowest order in α'. The generic cubic equation for the $\mathcal{H}_{\bar{1}2\bar{3}}$ component of the three-form is

$$h_1^3 + m h_1^2 + n h_1 + s = 0\,, \tag{4.62}$$

where m, n, s depend on h_4, h_2, t and α'. These are identified as

$$m = (a h_2 + b h_4)\,, \qquad n = \left(\frac{t^3}{\alpha'} + c h_4 h_2 + d h_4^2\right) \qquad \text{and} \qquad s = \left(-\frac{f t^3}{\alpha'} + e h_4^2 h_2\right),$$

where $a, b, \ldots$ are integers. These expressions can be easily determined, but for the analysis below we do not need the explicit form for these constants.

The equation above can now be written as (4.47) by using the shift technique for cubic equations. If we identify h_1 with $h - m/3$, then the p, q variables appearing in (4.47) can be given in terms of m, n and s as

$$p = n - \frac{m^2}{3} = \frac{t^3}{\alpha'} + B \qquad \text{and} \qquad q = s - \frac{mn}{3} + \frac{2m^3}{27} = -\frac{f t^3}{\alpha'} + D\,, \tag{4.63}$$

where B, D are *independent* of α'.

This is very important because it immediately implies, that to lowest order in α' the solution given in (4.55), (4.56) and (4.57) is still satisfied! Thus every component of $\mathcal{H}$ is proportional to the already determined solutions in (4.56) and (4.57). Therefore, we expect a similar potential for the radial modulus in the most generic case as well. This completes the argument.[17]

[17]The higher order terms are more involved and the simple analysis that we presented here gets modified. However we still expect to see similar behavior for *all* orders because the scaling arguments that we presented earlier continue to hold at every order and therefore the radial modulus gets fixed at any arbitrary order. We have not demonstrated this because the mathematics, though straightforward, becomes very involved at higher orders in α'. We hope to tackle this is future.

(*c*) **The $\mathcal{Q} = 0$ case.** So far we have discussed the case, when the cubic contribution $\tilde{\mathcal{H}}^3$ is non-zero. This is the usual case for some choices of holonomies. However, we could also have situations for which

$$\mathcal{Q} = \alpha_1^{ab}\ \alpha_2^{cd}\ \alpha_{\bar{3}}^{ef}\ \mathrm{Tr}(M^{ab}M^{cd}M^{ef}) = 0\,. \tag{4.64}$$

In this case, the cubic equation simplifies very much and to lowest order in α' we have $h = -q/p$. In fact, the relations for the Chern-Simons form will change to a much simpler relation between the spin-connection ω and the background three-form $\mathcal{H}$ as

$$\begin{aligned}\Omega_3(\tilde{\omega}) &= \tilde{\omega} \wedge d\tilde{\omega},\\ \Omega_3(\widehat{\omega}) &= \widehat{\omega} \wedge d\widehat{\omega}\,,\end{aligned} \tag{4.65}$$

where now comparing to the earlier expressions of Ω_3 we see that the forms are similar (in the usual case the form of $\Omega_3(\widehat{\omega})$ differs from $\Omega_3(\tilde{\omega})$).

Let us consider now a simpler situation in which we ignore the spin connection ω. In this case, we can show that the three-form equation will eventually be

$$\mathcal{H} - \frac{\alpha'}{4}\tilde{\mathcal{H}} \wedge d\tilde{\mathcal{H}} = f\,, \tag{4.66}$$

where f is defined as before, but in this case it has no ω dependence. A brief reflection shows us, that this case is more complicated, than the $\mathcal{Q} \neq 0$ case. Of course, to the zeroth order in α' we simply obtain $\mathcal{H} = f$, which is the real root obtained earlier. Having $\mathcal{Q} = 0$ means, that there are no terms of order α' and therefore the next non-trivial equation is of order α'^2. This is simply because (2.47) takes now the form

$$d\mathcal{H} = \alpha'\left[d\omega \wedge d\omega - d\omega \wedge d\tilde{\mathcal{H}} + \frac{1}{4}d\tilde{\mathcal{H}} \wedge d\tilde{\mathcal{H}} - \mathrm{tr}F \wedge F\right]. \tag{4.67}$$

This makes the situation a little involved and therefore (4.66) can only be solved iteratively. However, in the next sections we will not consider this case. To the zeroth order it is clear, that the only root is the real one and the second Chern class for the gauge bundle satisfies $c_2(F) = \int d\omega \wedge d\omega$. It is also clear, that even for this case the radius would be stabilized, because from (4.67) we see, that the left hand side scales as t, whereas the right hand side scales as $a\,t^{-1} + b\,t^{-2}$, where a, b can be easily ascertained from (4.67).[18]

(*d*) **Superpotential for the heterotic theory and radius of the six manifold.** In the above sections we argued, how a *complex* three-form could arise in the heterotic theory. In this section we would like to compare the superpotential, that we get from this setup to the superpotential obtained by doing U-dualities from the type-IIB theory. In order to do this, we must first incorporate back the spin connection ω into the calculations.

The main equation, which takes into account all the variables has already been spelled out in (4.42). Let us concentrate on the component h_1 for the time being. Using the earlier

[18]Furthermore in (4.67) we see that even though ω term fails to give a potential (discussed earlier in (4.4)) the term with $d\tilde{\mathcal{H}}$ could in principle be responsible for the potential. We will address this issue in more detail elsewhere.

definitions, we can show, that the cubic equation, that we get at this stage is the same as (4.62) with m, n, s defined as

$$m = a_1 + a_2\omega\,, \qquad n = \frac{t^3}{\alpha'} - a_3\omega + a_4\left(d\omega + \omega^2\right) + a_5 \qquad \text{and} \qquad s = -\frac{ft^3}{\alpha'} + \mathcal{O}\left(\omega^4\right), \tag{4.68}$$

where ω is the one-form spin connection ω_μ^{ab} and a_i are constants independent of α'. We can now define the shift β in $h_1 = h - \beta$ as $\beta = \frac{1}{3}(a_1 + a_2\omega)$. This results, as before, in a cubic equation of the form (4.47) with p and q given by

$$\begin{aligned} p &= \frac{t^3}{\alpha'} + \mathcal{A}\,\omega + \mathcal{O}(\omega^2)\,, \\ q &= -\frac{f\,t^3}{\alpha'} + \mathcal{B}\omega + \mathcal{O}(\omega^2)\,, \end{aligned} \tag{4.69}$$

where the α' dependence is shown explicitly and $\mathcal{A}$, $\mathcal{B}$ can be determined from (4.68). All the arguments dealt in the earlier sub-sections will go through without any subtlety. In particular $A \pm B$ are defined as usual and from there we can extract the three-form h_1 explicitly. The superpotential will now have the generic form, to lowest order in α'

$$W = \pm \int \left(f + ib\,\omega + \sum_{m,n\,\in\,\mathbb{Z}/2} i\,c_{mn}\,f^m t^n \right) \wedge \Omega\,, \tag{4.70}$$

where b depends on t in some specific way (the detailed dependence is not very important for our present purpose) and c_{mn} are constants that depend on α'.[19] Comparing the U-duality results of [28] we can see, that this form of the superpotential is what we expect, at least to this order. The shift of the superpotential proportional to the spin connection ω is, what is responsible for the twisting, as we mentioned earlier. This aspect has also been pointed out in [28] and also in the second reference of [10], where the detailed derivation of the superpotential using T-duality rules for the heterotic theory is given to the lowest order in α'. The higher powers of t terms in the superpotential are responsible for fixing the radial modulus of our manifold, as we show below. Therefore, to all orders in α' the superpotential will have the generic form $W = \pm\int \mathcal{H} \wedge \Omega$, with $\mathcal{H}$ given by

$$\mathcal{H} = \sum_{m,n,p,q\,\in\,\mathbb{Z}/2} b_{mnpq}\,\alpha'^m f^n (-t)^p \omega^q\,, \tag{4.71}$$

where f, t, ω are arranged (for every term) so that they are dimensionally same as f as we saw before and b_{mnpq} are constants. To obtain these terms from the type-IIB theory using T-duality, we need to look for higher order corrections to the T-duality rules. It will be interesting to do this, but this is at present beyond the scope of this work.

[19] There is yet another real contribution to the superpotential which appears from the heterotic gauge field F as $F \wedge J \wedge J$ where J is the fundamental two form for the manifold. This is distinct from $\Omega_3(A) \wedge \Omega$ where $\Omega_3(A)$ is the gauge Chern-Simons term. This term will be responsible for generating the Donaldson-Uhlenbeck-Yau kind of equations for gauge bundles. Furthermore the choice of the superpotential does indeed reproduce the torsional equation as is shown in [36]. More details on the superpotential and how to see the masses of the KK monopoles etc. will be addressed in part II of this paper.

Finally, we should use all the results derived above, to estimate the radius of our six-manifold. Again, we will concentrate only to the lowest order in α'. The potential for the generic case is the same as discussed in (4.59) except, that for our purpose we need to go to higher orders in t. We can write the potential as

$$V(t) = \frac{t^3}{\alpha'} + \frac{9}{64}\,\frac{\alpha'|f|^4}{t^3} + \mathcal{O}\left(\alpha'^2\right), \tag{4.72}$$

which is basically derived from (4.55), by keeping terms to order z^2. Recall also, that we are taking the superpotential as in (4.70). Minimizing $V(t)$ with respect to t, gives an estimate of the radius of the six-dimensional non-Kähler manifold. We can write this explicitly as

$$t = \left(\frac{3\alpha'|f|^2}{8}\right)^{1/3}, \tag{4.73}$$

where f is defined in (4.48). Observe also that, since $|f|$ can be determined directly from (2.7), one can numerically estimate the radius t of our manifold.

Before closing this section let us tie some loose ends. In deriving the radius for our manifold, we have expanded functions with respect to the quantity $1/p$, where p is defined in (4.48). From (4.73) we see, that this is possible, if the flux *density* f is large (with a total constant flux). Therefore, we need the radius t to be fixed but large. The limit of fixed but large enough radius is also consistent with the supergravity description. In fact, in the analysis done in section 3.1 of [6], we imposed

$$g_{het}^{(4)} \to 0, \qquad \text{and} \qquad V_{het} \gg 1, \tag{4.74}$$

in order to have a valid supergravity description. Here $g_{het}^{(4)}$ is the four dimensional heterotic coupling and V_{het} is the volume of the six-manifold. However, from (2.10) the reader may wonder, whether such choice is generically possible, because of the radius stabilization. But since all Kähler moduli are not fixed by our fluxes, we can constrain the volume of four-cycles in the original $\mathcal{M}$-theory picture to be small. Further investigations on this matter will be presented elsewhere.

There is yet another reason to have a large but finite radius. The vanishing of the gaugino and the gravitino supersymmetry transformations (in the notations of equation 4.29 of [6]) implies that

$$\nabla_M(\widehat{\omega})\,\epsilon = 0 = F^a_{MN}\Gamma^{MN}\epsilon, \tag{4.75}$$

converting (2.29) to the light-cone RNS (0,1) non-linear sigma model. This identification is precise *only*, when $d\mathcal{H} = 0$. However, we do not have this situation and therefore, we require the corrections to the set of equations (4.75) to be small. This is possible, if t^2/α' becomes sufficiently large. The above equation (4.75) can also be viewed as the condition under which (2.29) is invariant under world-sheet supersymmetry transformations on X^i, S^p. Therefore, when we identify (2.29) with the (0,1) sigma modeltransformations, we are also inherently assuming, that the corrections are small. Notice also, that the connection appearing in (4.75) is $\widehat{\omega}$ and therefore, the fact that the two-form J_{ij} is covariantly constant, is with respect to this connection. This aspect has been briefly alluded to in the earlier sections and discussed in much detail in a series of papers by Hull [2]. Furthermore, having $d\mathcal{H} \neq 0$ also means that we have the identity [2]

$$\frac{1}{2}\left[R_{abcd}(\tilde{\omega}) - R_{cdab}(\widehat{\omega})\right] = \mathcal{H}_{[abc,d]}\,, \tag{4.76}$$

implying a non-zero two-loop contribution to the trace anomalies.[20] Again, if t^2/α' is large, this contribution becomes small [37]. A non-vanishing Riemann tensor in (4.76) can again contribute to the beta function at two-loop order. This contribution has been calculated in many earlier works and is shown to be proportional to $R_{abcd}R^{abcd} - F_{ab}F^{ab}$. This would obviously cancel under the usual embedding of the spin connection into the gauge connection. For our case, we can argue, that this is suppressed, if the radius t is large. More details on this and phenomenological implications of large but finite radius will be explored in an upcoming paper [20].

5. Discussions

In this paper we studied in detail the geometrical and topological properties of non-Kähler manifolds of the form (2.7) and a large class of generalizations thereof. As we showed, these manifolds in general have zero Euler characteristics and also zero first Chern class. For the background studied earlier in [6], the complete topological properties were determined. The torsional metric in the presence and absence of gauge fields has been worked out and was shown in both cases to satisfy the torsional equations imposed by supersymmetry. In section 3 more general examples of non-Kähler manifolds were found and their mathematical properties determined. All these examples are compact and complex. In section 4 we determined the superpotential for compactifications of the heterotic string on such non-Kähler manifolds. We showed, that many of the moduli fields of these heterotic compactifications can be stabilized, once the $\mathcal{H}$-fluxes are turned on. In particular, we have computed the potential for the radial modulus and showed, that the value of this field can be determined.

5.1 Related examples of non-Kähler manifolds

Recently there have been some more examples of these manifolds discussed in the literature. One particular interesting one is the Iwasawa manifold first discussed in this context by Strominger [1] and more recently discussed in more detail by Cardoso et. al [7]. In fact, this example follows directly from the generic construction that we gave in section 3. The Iwasawa manifold is a principle torus bundle over a torus constructed from a set of 3×3 matrices with complex entries

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}, \tag{5.1}$$

[20]Recall, that the Ricci tensor is $R_{ab} \equiv R^c{}_{acb}(\tilde{\omega})$ with respect to the connection $\tilde{\omega}$. Being non-zero and finite, this contributes to the trace anomalies.

and is therefore, a complex manifold. As discussed in [1] and [7], if we restrict this to integers m, n, p and induce the action:

$$a \to a+m\,, \qquad b \to b+cm+n\,, \qquad c \to c+p\,, \tag{5.2}$$

then, we get a smooth manifold called the Iwasawa manifold. Therefore, the complete properties of heterotic string compactifications on the Iwasawa manifold can be determined. The background geometry is

$$\begin{aligned} ds^2 &= dzd\bar{z} + dvd\bar{v} + |du - zdv|^2, \quad e^{\phi} = c_o\,, \quad \chi = 0\,, \\ \mathcal{H} &= -\frac{1}{4}(du - zdv) \wedge d\bar{z} \wedge d\bar{v} + \text{c.c.}\,, \qquad \{b_i\} = (1,4,8,10,8,4,1)\,, \end{aligned} \tag{5.3}$$

with the anomaly condition $d\mathcal{H} = -\text{tr}\ F \wedge F$, solved with just an abelian gauge field configuration. The reader can extract more details on this from [7]. It will also be interesting to see, whether the Iwasawa manifold can appear from a four-fold in $\mathcal{M}$-theory in the same way as in [4, 6]. It is clear, that the naive identification of the four-fold as $T^4/\mathcal{I}_4 \times T^4$ cannot work because of two obvious reasons:

(a) The Euler characteristics of this four-fold is zero and therefore cannot support fluxes, as the anomaly constraints will prohibit it [5].

(b) The choice of $T^4/\mathcal{I}_4$ will tell us, that on the type-I side we should always have three-forms, that have one leg along the z^3 or $\bar{z}^3$ direction. From the explicit background constructed in [7] we see, that there are other components of $\mathcal{H}$.

Another detailed study of the mathematical aspects of non-Kähler manifolds has appeared recently in [9]. The manifolds considered there are of the form

$$ds^2 = e^{2\phi}\, g_{CY} + (dx + \alpha)^2 + (dy + \beta)^2\,, \tag{5.4}$$

where ϕ is the warp factor and g_{CY} is a Calabi-Yau base. These class of examples are also related to our construction. The one-forms α and β are defined on the Calabi-Yau base. These one-forms can be identified with (1,1) anti-self-dual forms ω_p and ω_q via $d\alpha = \omega_p$ and $d\beta = \omega_q$, which give rise to the three-form

$$H_3 = dx \wedge \omega_p + dy \wedge \omega_q\,, \tag{5.5}$$

in the type-IIB theory, after we make two T-dualities along the two-cycles of the fiber torus T^2. This three-form is basically the NS-NS three-form of the type-IIB theory and as such lies in the integer cohomology.

The paper [8] gave examples of new manifolds, that are neither complex nor Kähler in the context of the type-IIA theory. These manifolds are termed *half-flat*, and were shown to have torsion lying in all the five torsion classes $\mathcal{W}_i$. They generically have an SU(3) structure and could be complex, if the torsion lies in $\mathcal{W}_3 \oplus \mathcal{W}_4 \oplus \mathcal{W}_5$. The vanishing of the Nijenhuis tensor amounts to having, in this language, vanishing $\mathcal{W}_1 \oplus \mathcal{W}_2$ classes. This classification of torsion classes is well known in the mathematics literature but in connection to string theory compactifications it has also been addressed in [7]. The previous type-IIA models are mirrors of the type-IIB theory compactified on Calabi-Yau three-folds with NS-NS fluxes turned on. However, in [8] not all fields in the type-IIB theory are given an expectation value. In fact, one has to turn on at least the RR three-form to completely embed this in string theory. This has been done recently in [28]. The authors of [28] gave explicit examples of mirror manifolds in the type-IIA theory and showed, that these manifolds have an almost complex structure, which may or may-not be integrable. When it is integrable, then the manifold is complex. The model discussed in [28] is the type IIB theory on $T^6/\mathbb{Z}_2$, and therefore making less than six T-dualities we always remain in either type-IIA or type-IIB (depending on whether we make an odd or even number of T-dualities). Hence, we have mirror descriptions in either of these theories on generic compact non-Kähler manifolds with an almost complex structure J_{mn}. For the models presented in [8] the fundamental two form J_{mn} is not covariantly constant with respect to the affine connection but *is* covariantly constant with respect to the preferred connection measured by the contorsion tensor. In the language of ours the contorsion tensor is precisely $\frac{1}{2}\mathcal{H}$ and therefore the connection can be identified with the $\widehat{\omega}$ discussed in the present paper.

Further discussions of type-IIB compactifications on six-dimensional manifolds with fluxes have been recently discussed in [10].

5.2 Phenomenological applications

The models considered herein might be rather interesting for particle phenomenology, as many moduli fields appearing in these compactifications (including the radial modulus) can be stabilized and definite predictions for the coupling constants of the standard model can be made. Furthermore, our compactifications include a warp factor, which provides one of the few known mechanisms for solving the gauge hierarchy problem [11]. Recently it has been speculated that the masses generated by the fluxes could be even at the phenomenological viable TeV scale [38].

Another important property of our compactification manifolds is, that they have zero Euler characteristics and a vanishing first Chern class. One may wonder, if a vanishing Euler number implies a vanishing number of particle generations in the four-dimensional theory. It is well known [39], that the net number of generations minus anti-generations is determined in terms of the Euler number of the internal manifold as

$$|h^{2,1} - h^{1,1}| = \frac{|\chi|}{2}\,, \tag{5.6}$$

for compactifications on Calabi-Yau three-folds, where $h^{i,j}$ describe the corresponding Hodge numbers of the internal manifold. However, the above formula is not valid for the models considered herein, as the spin connection has to be embedded into the gauge connection to arrive at the above result, even for the case of Kähler compactifications. This is not the case we are interested in. To determine the number of generations appearing in our models, we would need to analyze the zero modes of the Dirac equation for our backgrounds. We shall report this in a future publication [20].

In this paper we have shown, that the radial modulus for compactifications of the heterotic string on non-Kähler manifolds receives a potential, which allowed us to estimate the actual value of the size of the internal manifold. A tantalizing possibility would be, that a cosmological constant is indeed induced, after the radial modulus has been stabilized. A priori, we see here the possibility, that a positive cosmological constant could be induced, giving us a realization of de Sitter space in string theory. This is a long standing puzzle, whose solution could certainly be along these lines. We leave the details of this fascinating possibility for future work [20]. It would be a great triumph of string theory, if the correct relation between the supersymmetry breaking scale and the cosmological constant could be predicted in this way [40].

5.3 Future directions

- There are many interesting directions to pursue in the future. In the sigma-model section we discussed the fact, that for our case the simplest embedding of the gauge-connection into the torsional-spin connection is not allowed. Therefore, there might be the possibility, that the two-loop beta function is not vanishing. As discussed earlier and also in [41] and [2], these contributions are suppressed, if the size of the six-manifold is large. In fact, for our case we can have a large sized manifold, as the size parameter depends on the flux-density. Furthermore, the background found in [6] by using U-dualities from a given supergravity background in $\mathcal{M}$-theory, is a valid solution at least to the *lowest* order in α'. Therefore, (to extend the discussion to all orders in α') two things could happen here:

  (a) Apart from the size factor, there could be generic counter-terms, that could cancel the two loop contributions to the beta function. At least it has been discussed in some detail in the literature, that this contribution is cancelled, if we define the background three-form $\mathcal{H}$ as in (2.47), because there exist counter-terms [41] and [2], that lead to a vanishing beta function. But this relies on the fact, that we are embedding the gauge connection into the torsional-spin connection $\tilde{\omega}$, which is not what we want to do in the present case.

  (b) The beta function is exactly zero to all orders in α' only for a given size of the six manifold. This is what we might expect, because the radial modulus is fixed and therefore only, when the background has the right radial modulus and right complex structure modulus, the beta function for strings propagating on this background will be zero. This matter needs further investigation and more details will be presented elsewhere.

- In this paper we studied SO(32) heterotic strings on compact non-Kähler manifolds or more appropriately D_4^4 heterotic strings. It would be interesting to see, how to describe the $E_8 \times E_8$ heterotic string in this framework. Observe, that the reason we have gotten the D_4^4 heterotic theory is, because we started with an orientifold model in the type-IIB theory, that under two T-dualities and an S-duality reproduces the D_4^4 theory [6]. To get the $E_8 \times E_8$ heterotic string we need a similar framework, maybe in $\mathcal{F}$-theory, where there is a possibility of having exceptional symmetries [42]. But

JHEP04(2003)007

the points, where we can have exceptional symmetries and orientifold representations are no longer *perturbative* [42]. Let us elaborate this a little bit. More details will appear in [20].

The models studied in [4] and [6] have an $\mathcal{F}$-theory interpretation defined in terms of elliptic curves

$$y^2 = x^3 + xf(z) + g(z)\,, \tag{5.7}$$

where z is the coordinate on the CP^1 base and f, g are polynomials of degree 8 and 12 respectively [43]. The modular parameter of the fiber is given in terms of j-function [43], where $j \propto \frac{f^3}{\Delta}$ and Δ (not to be confused with warp-factor) is the discriminant. The $\mathcal{F}$-theory representation, that concretely realizes the model of [6], has the following choice of f, g and Δ

$$f(z) \sim (z - z_1)^2\,, \qquad g(z) \sim (z - z_1)^3\,, \qquad \Delta(z) \sim (z - z_1)^6\,, \tag{5.8}$$

near one "orientifold" point $z \to z_1$. The fact that Δ is proportional to z^6 means, that all the orientifold planes have become dynamical realizing the D_4 symmetry. Now, as shown in [42], an E_8 symmetry is realized at $z \to z_1$, when f, g, Δ are

$$f(z) = 0\,, \qquad g(z) \sim (z - z_1)^5\,, \qquad \Delta(z) \sim (z - z_1)^{10}\,. \tag{5.9}$$

A couple of immediate points to note here are, that $K3$ goes to its Z_6 orbifold point (with full symmetry of $E_8 \times E_6 \times D_4$) and the fact that we have $g \sim z^5$ will tell us, from Tate's algorithm, that the symmetry is E_8. This immediately tells us, that the pure $E_8 \times E_8$ symmetry is realized, when

$$g(z) = (z - z_1)^5 (z - z_2)^5 (z - z_3)(z - z_4)\,, \tag{5.10}$$

with vanishing $f(z)$ as before. This has a non-perturbative orientifold representation, as was shown in [42]. A more concrete way of realizing this construction was discussed in [44]. Thus, this could be one possible way of getting the torsional background for the $E_8 \times E_8$ heterotic theory.

- The non-Kähler manifolds discussed in this paper all have a vanishing Euler characteristics. This is not a problem by itself, as we discussed above, that a vanishing Euler characteristics *doesn't* imply zero number of generations. It will be interesting to extend the above analysis to non-zero Euler characteristics. We would then have to start on the type-IIB side with a manifold with non-zero Euler characteristics in the *absence* of fluxes. Let us elaborate this a little bit. In the present paper we have seen, that in the absence of NS-NS and R-R fluxes in the type-IIB picture, making two T-dualities we end up in the type-I theory on $K3 \times T^2$. As discussed earlier, by switching on fluxes, the torus T^2 becomes nontrivially fibred over the base $K3$. However $K3 \times T^2$ has zero Euler characteristics (because of the T^2) and therefore, the non-Kähler manifold of the heterotic compactification also has zero Euler characteristics. For the generalization of the present construction to non-zero Euler characteristics, we should start with a manifold, which looks like $K3 \times Z$, where Z is a two-dimensional manifold with non-zero Euler characteristics on the type-IIB side. This is the minimal requirement. Of course, we can even get a generic six-dimensional manifold X, which should then have the following properties in the *absence* of fluxes: (a) compact and complex with non-zero Euler characteristics, (b) there exists a four-fold, which is a non-trivial T^2 fibration over X, and most importantly (c) should have an orientifold setup in the type-IIB framework. More details on this will be addressed in a future publication [20].

- In subsection 2.5 we discussed the anomaly relation for the heterotic theory and showed, how the tr $F \wedge F$ term can appear in the heterotic Bianchi identity from the type IIB side. The allowed gauge bundle is very restricted for the torsional background, because, as for the Kähler compactifications, we expect the gauge bundle to satisfy

$$g^{a\bar{b}} F^n_{a\bar{b}} = 0\,, \qquad F^n_{ab} = 0\,, \qquad F^n_{\bar{a}\bar{b}} = 0\,, \tag{5.11}$$

which are the Donaldson-Uhlenbeck-Yau (DUY) equations for gauge fields. It would be interesting to see, how the DUY constraints can be derived from a D-term along the lines of [39] and [27] for the case of compactifications on Calabi-Yau three-folds . In [6] it was shown, how in the type-IIB theory this is realized from the primitivity condition of G-fluxes in $\mathcal{M}$-theory. Furthermore, because the three-form $\mathcal{H}$ is related to the metric by the torsional equations (4.38), there arises another restriction on the allowed gauge bundle

$$\text{Tr } F \wedge F = 30 \left[\text{tr } R \wedge R - \frac{i}{\alpha'}\, \partial\bar{\partial} J \right]. \tag{5.12}$$

Since $\mathcal{H}$ is globally defined, the constraint (2.52) follows from this formula. The relation (5.12) is non-trivial, because none of the terms in the right hand side can be zero. Thus, the properties of the gauge bundle is another important aspect, that needs to be studied in a future [20].

- We discussed in some detail, how many of the moduli for the heterotic theory, including the radial modulus, can be fixed for this kind of compactification. In the earlier sections we argued, that the dilaton modulus does not get fixed in this process. One might argue, that since the superpotential in type-IIB theory has an axion-dilaton appearing explicitly in the formula, the heterotic superpotential, which follows from a set of U-dualities, should also have a dilaton dependence. This is not quite so, because after performing two T-dualities and an S-duality on the type-IIB superpotential one can show, using the T-duality rules of [21], that the dilaton factor *does not* appear in the formula for the heterotic superpotential. Therefore, it will be interesting to see, how explicitly the dilaton could appear in the heterotic superpotential. Fixing the dilaton will immediately guarantee, that we can have a constraint on the size of the six manifold in the type-IIB framework. This would imply, that a string theory model

JHEP04(2003)007

in de-Sitter space could be constructed, along the lines of the first paper of [45].[21] Furthermore, as discussed in the other papers of [45], the type-IIB models with fluxes and controlled moduli give the supergravity dual of cascading $\mathcal{N}=1$ gauge theories, which are confining in the IR. A better understanding of the moduli problem in the type-IIB setup can therefore be used, to understand the dynamics of $\mathcal{N}=1$ gauge theories.

- In the examples studied in section 4, we have considered the case, when the preferred spin connection is not embedded into the gauge connection and the relation between A and $\tilde{\omega}$ is given by (2.49). This in particular implies, that we have

$$d\mathcal{H} = d\tilde{\mathcal{H}} = \mathcal{O}(\alpha')\,, \tag{5.13}$$

  and therefore in section 4.4 had a simple way to study the stabilization of the radial modulus. This is not always the case and, in fact, for a very generic embedding we can have a situation, where the only possible way to study the three-form is by iterative technique. However, it is clear by scaling arguments, that the radius is again fixed for this case (see sec 4.4(c)). It will therefore be interesting to see, what value of the radius we get in this generic situation.

Acknowledgments

It is our pleasure to thank Michael Dine, Lance Dixon, Edward Goldstein, Shamit Kachru, Joe Polchinski, Sergey Prokushkin, Savdeep Sethi, Mohammed M. Sheikh-Jabbari, Andrew Strominger, Prasanta Tripathy, Sandip Trivedi and Edward Witten for useful discussions. The work of K.B. was supported by the University of Utah. The work of M. B. is supported by NSF grant PHY-01-5-23911 and an Alfred Sloan Fellowship. The work of K.D. was supported in part by a David and Lucile Packard Foundation Fellowship 2000 − 13856. K.B. wishes to thank the warm hospitality at the University of Maryland, where part of this work has been carried out.

References

[1] A. Strominger, *Superstrings with torsion*, *Nucl. Phys.* **B 274** (1986) 253.

[2] C.M. Hull, *Superstring compactifications with torsion and space-time supersymmetry*, in proceedings, *Superunification and extra dimensions*, Turin 1985, pp. 347–375; *Sigma model beta functions and string compactifications*, *Nucl. Phys.* **B 267** (1986) 266; *Compactifications of the heterotic superstring*, *Phys. Lett.* **B 178** (1986) 357; *Lectures on nonlinear sigma models and strings*, lectures given at super field theories workshop, Vancouver, Canada 1986.

[3] B. de Wit, D.J. Smit and N.D. Hari Dass, *Residual supersymmetry of compactified $D=10$ supergravity*, *Nucl. Phys.* **B 283** (1987) 165.

[21] We have been informed, that some stable de-Sitter solutions have recently been found in type-IIB theory with fluxes [46].

[4] K. Dasgupta, G. Rajesh and S. Sethi, *M-theory, orientifolds and G-flux*, *J. High Energy Phys.* **08** (1999) 023 [hep-th/9908088].

[5] K. Becker and M. Becker, *calM-theory on eight-manifolds*, *Nucl. Phys.* **B 477** (1996) 155 [hep-th/9605053].

[6] K. Becker and K. Dasgupta, *Heterotic strings with torsion*, *J. High Energy Phys.* **11** (2002) 006 [hep-th/0209077].

[7] G.L. Cardoso et al., *Non-Kähler string backgrounds and their five torsion classes*, *Nucl. Phys.* **B 652** (2003) 5 [hep-th/0211118].

[8] S. Gurrieri, J. Louis, A. Micu and D. Waldram, *Mirror symmetry in generalized Calabi-Yau compactifications*, *Nucl. Phys.* **B 654** (2003) 61 [hep-th/0211102].

[9] E. Goldstein and S. Prokushkin, *Geometric model for complex non-kaehler manifolds with* SU(3) *structure*, hep-th/0212307.

[10] S. Gurrieri and A. Micu, *Type-IIB theory on half-flat manifolds*, hep-th/0212278;
P.K. Tripathy and S.P. Trivedi, *Compactification with flux on $K3$ and tori*, hep-th/0301139.

[11] S.B. Giddings, S. Kachru and J. Polchinski, *Hierarchies from fluxes in string compactifications*, *Phys. Rev.* **D 66** (2002) 106006 [hep-th/0105097];
S. Kachru, M.B. Schulz and S. Trivedi, *Moduli stabilization from fluxes in a simple IIB orientifold*, hep-th/0201028;
A.R. Frey and J. Polchinski, *$N = 3$ warped compactifications*, *Phys. Rev.* **D 65** (2002) 126009 [hep-th/0201029].

[12] D.J. Gross, J.A. Harvey, E.J. Martinec and R. Rohm, *The heterotic string*, *Phys. Rev. Lett.* **54** (1985) 502; *Heterotic string theory, 1. The free heterotic string*, *Nucl. Phys.* **B 256** (1985) 253; *Heterotic string theory, 2. The interacting heterotic string*, *Nucl. Phys.* **B 267** (1986) 75.

[13] C. M. Hull and E. Witten, *Supersymmetric sigma models and the heterotic string*, *Phys. Lett.* **B 160** (1985) 398;
A. Sen, *Local gauge and Lorentz invariance of the heterotic string theory*, *Phys. Lett.* **B 166** (1986) 300; *The heterotic string in arbitrary background field*, *Phys. Rev.* **D 32** (1985) 2102; *Equations of motion for the heterotic string theory from the conformal invariance of the sigma model*, *Phys. Rev. Lett.* **55** (1985) 1846.

[14] S. Ivanov and G. Papadopoulos, *A no-go theorem for string warped compactifications*, *Phys. Lett.* **B 497** (2001) 309 [hep-th/0008232].

[15] C.M. Hull, *Anomalies, ambiguities and superstrings*, *Phys. Lett.* **B 167** (1986) 51.

[16] E. Witten, *Some properties of $O(32)$ superstrings*, *Phys. Lett.* **B 149** (1984) 351.

[17] M. Dine, N. Seiberg, X. G. Wen and E. Witten, *Nonperturbative effects on the string world sheet*, *Nucl. Phys.* **B 278** (1986) 769; *Nonperturbative effects on the string world sheet, 2*, *Nucl. Phys.* **B 289** (1987) 319.

[18] M. Dine and N. Seiberg, *Couplings and scales in superstring models*, *Phys. Rev. Lett.* **55** (1985) 366.

[19] G. W. Gibbons and P.J. Ruback, *The hidden symmetries of multicenter metrics*, *Commun. Math. Phys.* **115** (1988) 267;
N.S. Manton and B.J. Schroers, *Bundles over moduli spaces and the quantization of BPS monopoles*, *Ann. Phys. (NY)* **225** (1993) 290;

A. Sen, *Dyon-monopole bound states, selfdual harmonic forms on the multi-monopole moduli space and* SL(2, $\mathbb{Z}$) *invariance in string theory*, *Phys. Lett.* **B 329** (1994) 217 [hep-th/9402032].

[20] K. Becker, M. Becker, K. Dasgupta and P.S. Green, work in progress.

[21] P. Meessen and T. Ortin, *An* SL(2, $\mathbb{Z}$) *multiplet of nine-dimensional type-II supergravity theories*, *Nucl. Phys.* **B 541** (1999) 195 [hep-th/9806120];
E. Bergshoeff, C.M. Hull and T. Ortin, *Duality in the type-II superstring effective action*, *Nucl. Phys.* **B 451** (1995) 547 [hep-th/9504081];
S.F. Hassan, *T-duality, space-time spinors and RR fields in curved backgrounds*, *Nucl. Phys.* **B 568** (2000) 145 [hep-th/9907152].

[22] A. Sen, *Strong coupling dynamics of branes from M-theory*, *J. High Energy Phys.* **10** (1997) 002 [hep-th/9708002];
K. Dasgupta, D.P. Jatkar and S. Mukhi, *Gravitational couplings and Z_2 orientifolds*, *Nucl. Phys.* **B 523** (1998) 465 [hep-th/9707224];
J.F. Morales, C.A. Scrucca and M. Serone, *Anomalous couplings for D-branes and o-planes*, *Nucl. Phys.* **B 552** (1999) 291 [hep-th/9812071];
B. Stefański Jr., *Gravitational couplings of D-branes and O-planes*, *Nucl. Phys.* **B 548** (1999) 275 [hep-th/9812088].

[23] A. Sen, *A note on enhanced gauge symmetries in M- and string theory*, *J. High Energy Phys.* **09** (1997) 001 [hep-th/9707123].

[24] P. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley, New York 1978.

[25] K. Becker, M. Becker, M. Haack and J. Louis, *Supersymmetry breaking and alpha'-corrections to flux induced potentials*, *J. High Energy Phys.* **06** (2002) 060 [hep-th/0204254].

[26] J. Bagger and J. Wess, *Supersymmetry and supergravity*, Princeton University Press, Princeton 1983.

[27] M. Becker and D. Constantin, *A note on flux induced superpotentials in string theory*, hep-th/0210131.

[28] S. Kachru, M.B. Schulz, P.K. Tripathy and S.P. Trivedi, *New supersymmetric string compactifications*, hep-th/0211182.

[29] E. Silverstein, *(A)dS backgrounds from asymmetric orientifolds*, hep-th/0106209;
A. Dabholkar and C. Hull, *Duality twists, orbifolds, and fluxes*, hep-th/0210209;
S. Hellerman, J. McGreevy and B. Williams, *Geometric constructions of nongeometric string theories*, hep-th/0208174.

[30] K.S. Narain, *New heterotic string theories in uncompactified dimensions* < 10, *Phys. Lett.* **B 169** (1986) 41;
K.S. Narain, M.H. Sarmadi and E. Witten, *A note on toroidal compactification of heterotic string theory*, *Nucl. Phys.* **B 279** (1987) 369;
J. Maharana and J.H. Schwarz, *Noncompact symmetries in string theory*, *Nucl. Phys.* **B 390** (1993) 3 [hep-th/9207016].

[31] N. Kaloper and R.C. Myers, *The O(dd) story of massive supergravity*, *J. High Energy Phys.* **05** (1999) 010 [hep-th/9901045];
G. Curio, A. Klemm, B. Körs and D. Lüst, *Fluxes in heterotic and type-II string compactifications*, *Nucl. Phys.* **B 620** (2002) 237 [hep-th/0106155];
J. Louis and A. Micu, *Heterotic string theory with background fluxes*, *Nucl. Phys.* **B 626** (2002) 26 [hep-th/0110187].

[32] V. Kaplunovsky, J. Louis and S. Theisen, *Aspects of duality in $N = 2$ string vacua*, *Phys. Lett.* **B 357** (1995) 71 [hep-th/9506110].

[33] N. Seiberg, *Observations on the moduli space of superconformal field theories*, *Nucl. Phys.* **B 303** (1988) 286;
A. Ceresole, R. D'Auria, S. Ferrara and A. Van Proeyen, *Duality transformations in supersymmetric Yang-Mills theories coupled to supergravity*, *Nucl. Phys.* **B 444** (1995) 92 [hep-th/9502072].

[34] E. Cremmer, S. Ferrara, L. Girardello and A. Van Proeyen, *Yang-Mills theories with local supersymmetry: lagrangian, transformation laws and superhiggs effect*, *Nucl. Phys.* **B 212** (1983) 413;
T.R. Taylor and C. Vafa, *RR flux on Calabi-Yau and partial supersymmetry breaking*, *Phys. Lett.* **B 474** (2000) 130 [hep-th/9912152].

[35] S. Gukov, C. Vafa and E. Witten, *CFT's from Calabi-Yau four-folds*, *Nucl. Phys.* **B 584** (2000) 69 [hep-th/9906070].

[36] K. Becker, M. Becker, K. Dasgupta and S. Prokushkin, *Properties of heterotic vacua from superpotentials*, hep-th/0304001.

[37] C.M. Hull and P.K. Townsend, *The two loop beta function for sigma models with torsion*, *Phys. Lett.* **B 191** (1987) 115; *World sheet supersymmetry and anomaly cancellation in the heterotic string*, *Phys. Lett.* **B 178** (1986) 187.

[38] O. DeWolfe and S.B. Giddings, *Scales and hierarchies in warped compactifications and brane worlds*, hep-th/0208123.

[39] E. Witten, *New issues in manifolds of* SU(3) *holonomy*, *Nucl. Phys.* **B 268** (1986) 79.

[40] T. Banks, *Cosmological breaking of supersymmetry or little lambda goes back to the future. II*, hep-th/0007146.

[41] C.G. Callan Jr., E.J. Martinec, M.J. Perry and D. Friedan, *Strings in background fields*, *Nucl. Phys.* **B 262** (1985) 593;
A. Sen, *Equations of motion for the heterotic string theory from the conformal invariance of the sigma model*, *Phys. Rev. Lett.* **55** (1985) 1846; *The heterotic string in arbitrary background field*, *Phys. Rev.* **D 32** (1985) 2102.

[42] K. Dasgupta and S. Mukhi, *F-theory at constant coupling*, *Phys. Lett.* **B 385** (1996) 125 [hep-th/9606044].

[43] C. Vafa, *Evidence for F-theory*, *Nucl. Phys.* **B 469** (1996) 403 [hep-th/9602022];
A. Sen, *F-theory and orientifolds*, *Nucl. Phys.* **B 475** (1996) 562 [hep-th/9605150];
T. Banks, M.R. Douglas and N. Seiberg, *Probing F-theory with branes*, *Phys. Lett.* **B 387** (1996) 278 [hep-th/9605199].

[44] M.R. Gaberdiel and B. Zwiebach, *Exceptional groups from open strings*, *Nucl. Phys.* **B 518** (1998) 151 [hep-th/9709013].

[45] K. Dasgupta, C. Herdeiro, S. Hirano and R. Kallosh, *$D3/D7$ inflationary model and M-theory*, *Phys. Rev.* **D 65** (2002) 126002 [hep-th/0203019];
K. Dasgupta, K.-h. Oh, J. Park and R. Tatar, *Geometric transition versus cascading solution*, *J. High Energy Phys.* **01** (2002) 031 [hep-th/0110050].

[46] S. Kachru, R. Kallosh, A. Linde and S.P. Trivedi, *De Sitter vacua in string theory*, hep-th/0301240.